

The de Almeida-Thouless line in vector spin glasses

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We consider the infinite-range spin glass in which the spins have $m > 1$ components (a vector spin glass). Applying a magnetic field which is random in direction, there is an Almeida Thouless (AT) line below which the "replica symmetric" solution is unstable, just as for the Ising ($m = 1$) case. We calculate the location of this AT line for Gaussian random fields for arbitrary m , and verify our results by numerical simulations for $m = 3$.

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I. INTRODUCTION

The infinite-range Ising spin glass, first proposed by Sherrington and Kirkpatrick¹, has been extensively studied. It was found by de Almeida and Thouless² (hereafter referred to as AT) that the simple "replica symmetric" (RS) ansatz for the spin glass state becomes unstable below a line in the magnetic field-temperature plane, known as the AT line. While the Ising spin has $m = 1$ components, the m -component vector spin glass for $m > 1$ has received less attention. de Almeida et al.³ (hereafter referred to as AJKT) found an instability in zero field, but did not consider the effects of a magnetic field. The effects of a *uniform* field on a vector spin glass were first studied by Gabay and Toulouse⁴. They found a line of transitions (the GT line), which is of a different nature from the AT line. In a uniform field, a distinction has to be made between spin components longitudinal and transverse to the field, and the GT line is the spin glass ordering of the transverse components, and these are effectively in *zero field*^{5,6}. The AT line is different from the Gabay-Toulouse⁴ (GT) line, since it is a transition to a phase with replica symmetry breaking but with *no change in spin symmetry*. The existence of the AT line is perhaps the most striking prediction of the mean field theory of spin glasses. The GT line occurs at a higher temperature than the putative AT line, which becomes simply a crossover^{5,6}.

The main point of the present work is to argue that one should consider not a uniform field but a field which is random in *direction* (it will also be convenient to make it random in magnitude though this is not essential) and that, in this case, there *is* an AT line also for vector spin glasses. We will determine the location of this line for an arbitrary number of spin components.

The Hamiltonian is given by

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_i \mathbf{h}_i \cdot \mathbf{S}_i, \quad (1)$$

where the S_i^μ , ($\mu = 1, \dots, m$) are m -component spins of length $m^{1/2}$, i.e.

$$\sum_{\mu=1}^m (S_i^\mu)^2 = m, \quad (2)$$

the interactions J_{ij} between all distinct pairs of spins $\langle i, j \rangle$ are independent random variables with zero mean and variance given by

$$[J_{ij}^2]_{\text{av}} = \frac{J^2}{N-1}, \quad (3)$$

and the h_i^μ are independent Gaussian random fields, uncorrelated between sites, with zero mean and which satisfy

$$[h_i^\mu h_j^\nu]_{\text{av}} = h_r^2 \delta_{ij} \delta_{\mu\nu}. \quad (4)$$

The notation $[\dots]_{\text{av}}$ indicates an average over the quenched disorder. The normalization of the spins in Eq. (2) is chosen so that the zero field transition temperature is

$$T_c = J \quad (5)$$

for all m .

Consider first the Ising case ($m = 1$). The spin glass order parameter is

$$q \equiv \frac{1}{N} \sum_i [\langle S_i \rangle^2]_{\text{av}}, \quad (6)$$

where $\langle \dots \rangle$ denotes a thermal average. From linear response theory, if we make small additional random changes, δh_i , in the random fields, uncorrelated with each other and the original values of the fields, the change in $\langle S_i \rangle$ is given by

$$\delta \langle S_i \rangle = \frac{1}{T} \sum_j \chi_{ij} \delta h_j, \quad (7)$$

where the linear response function χ_{ij} is given by

$$\chi_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle, \quad (8)$$

and, for convenience, we have separated out the factor of $1/T$. Hence the change in q is given by

$$\delta q = \frac{1}{T^2} \frac{1}{N} \sum_{i,j,k} [\chi_{ij} \chi_{ik}]_{\text{av}} [\delta h_j \delta h_k]_{\text{av}}, \quad (9)$$

$$= \frac{1}{T^2} \chi_{SG} \delta h_r^2, \quad (10)$$

where

$$\begin{aligned}\chi_{SG} &= \frac{1}{N} \sum_{i,j} [\chi_{ij}^2]_{\text{av}}, \\ &= \frac{1}{N} \sum_{i,j} [(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)^2]_{\text{av}}\end{aligned}\quad (11)$$

is the spin glass susceptibility.

The corresponding results for vector spins are easily obtained. The change in the spin glass order parameter,

$$q \equiv \frac{1}{N} \sum_i \frac{1}{m} \sum_{\mu} [(S_i^{\mu})^2]_{\text{av}}, \quad (12)$$

is given by

$$\delta q = \frac{1}{T^2} \frac{1}{N} \sum_{i,j,k} \frac{1}{m} \sum_{\mu,\nu,\eta} [\chi_{ij}^{\mu\nu} \chi_{ik}^{\mu\eta}]_{\text{av}} [\delta h_j^{\nu} \delta h_k^{\eta}]_{\text{av}}, \quad (13)$$

$$= \frac{1}{T^2} \chi_{SG} \delta h_r^2, \quad (14)$$

where now

$$\chi_{SG} = \frac{1}{N} \sum_{i,j} \frac{1}{m} \sum_{\mu,\nu} [(\chi_{ij}^{\mu\nu})^2]_{\text{av}}, \quad (15)$$

$$= \frac{1}{N} \sum_{i,j} \frac{1}{m} \sum_{\mu,\nu} [(\langle S_i^{\mu} S_j^{\nu} \rangle - \langle S_i^{\mu} \rangle \langle S_j^{\nu} \rangle)^2]_{\text{av}}. \quad (16)$$

For the Ising case, the sign of the field can be “gauged away” by the transformation $S_i \rightarrow -S_i$, and $J_{ij} \rightarrow -J_{ij}$ for all j . Hence the only difference between a uniform field and a Gaussian random field is that the latter varies in *magnitude*, and these magnitude fluctuations turn out to have only a minor effect⁷. However, for the vector case, the random *direction* of the Gaussian random field does make a big difference because there is no longer a distinction between longitudinal and transverse, and so there is *no longer a GT line to preempt the AT line*.

In zero field, χ_{SG} diverges at the transition temperature T_c given in Eq. (5), which is expected since χ_{SG} is the susceptibility corresponding to the order parameter. Surprisingly, AT showed for the Ising case ($m = 1$) that it also diverges in a magnetic field (either uniform, as originally considered by AT, or random, as considered later by Bray⁷) along the AT line in the field-temperature plane. Below the AT line, χ_{SG} goes negative, indicating that the RS solution is incorrect, and has to be replaced by the Parisi^{8,9} replica symmetry breaking (RSB) solution.

In this paper we calculate χ_{SG} for a vector spin glass in the presence of a random field, and show that it also becomes negative below an AT line in the h_r - T plane, whose location we calculate. This fact does not appear to be widely recognized. Although a field which is random in direction can presumably not be applied experimentally, we feel that there is theoretical interest in our result because a random field *can* be applied in simulations. Whether or not an AT line exists in finite-range

spin glasses, is a crucial difference between the replica symmetry breaking (RSB) picture^{8–11} of the spin glass state, where it does occur, and the droplet picture^{12–15}, where it does not. It has been found possible to simulate Heisenberg spin glasses for significantly larger sizes^{16–18} than Ising spin glasses, so our results may give an additional avenue through which to investigate numerically the nature of the spin glass state.

The plan of this paper is as follows. In Sec. II we compute the non-linear susceptibility for the Ising spin glass following the lines of AT. In Sec. III we do the corresponding calculation for the vector spin glass. This is followed in Sec. IV by a numerical evaluation of the AT line for several values of m and a confirmation of the results by Monte Carlo simulations for the Heisenberg spin glass, $m = 3$. We summarize our results in Sec. V. Many of the technical details are relegated to appendices.

II. THE SPIN GLASS SUSCEPTIBILITY FOR ISING SPIN GLASSES

In this section we review the calculation of the AT line for the Ising case. In the next section we shall use this approach to derive the AT line for vector spin glasses.

The standard way of averaging in random systems is the replica trick, which exploits the result

$$\ln Z = \lim_{n \rightarrow 0} \frac{Z^n - 1}{n}. \quad (17)$$

Applying this to the Ising ($m = 1$) version of the Hamiltonian in Eq. (1), one has

$$\begin{aligned}[Z^n]_{\text{av}} &= \text{Tr} \exp \left[\frac{(\beta J)^2}{2N} \sum_{\langle i,j \rangle} \sum_{\alpha,\beta} S_i^{\alpha} S_j^{\alpha} S_i^{\beta} S_j^{\beta} \right. \\ &\quad \left. + \frac{h_r^2}{2} \sum_i \sum_{\alpha,\beta} S_i^{\alpha} S_i^{\beta} \right].\end{aligned}\quad (18)$$

We denote averages over the effective replica Hamiltonian in the exponential on the RHS of Eq. (18) by $\langle \dots \rangle$. Following standard steps, see e.g. Refs. 1,19, one obtains (omitting an unimportant overall constant)

$$\begin{aligned}[Z^n]_{\text{av}} &= \int_{-\infty}^{\infty} \left(\prod_{(\alpha\beta)} \left(\frac{N}{2\pi} \right)^{1/2} (\beta J) dq_{\alpha\beta} \right) \\ &\quad \times \exp \left(-N \frac{(\beta J)^2}{2} \sum_{(\alpha\beta)} q_{\alpha\beta}^2 \right) \left(\text{Tr} \exp L[q_{\alpha\beta}] \right)^N,\end{aligned}\quad (19)$$

where $L[q_{\alpha\beta}]$ is given by

$$L[q_{\alpha\beta}] = \beta^2 \sum_{(\alpha\beta)} (J^2 q_{\alpha\beta} + h_r^2) S^{\alpha} S^{\beta}, \quad (20)$$

the trace is over the spins S^{α} , $\alpha = 1, \dots, n$, and $(\alpha\beta)$ denotes one of the $n(n-1)/2$ distinct pairs of replicas.

We take the replica symmetric (RS) saddle point, where all the $q_{\alpha\beta}$ are equal to the same value q . The spin traces at the RS saddle point are evaluated by writing

$$\begin{aligned} \text{Tr } e^L &= \text{Tr} \exp \left(\beta^2 \sum_{(\alpha\beta)} (J^2 q + h_r^2) S^\alpha S^\beta \right) \\ &= \text{Tr} \exp \left(\frac{\beta^2}{2} (J^2 q + h_r^2) \left(\left\{ \sum_\alpha S^\alpha \right\}^2 - n \right) \right) \\ &\propto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz \prod_{\alpha=1}^n \left[\text{Tr} e^{\beta(J^2 q + h_r^2)^{1/2} z S^\alpha} \right], \end{aligned} \quad (21)$$

where, in the last line, we omitted the constant factor $\exp[-(\beta^2/2)(J^2 q + h_r^2)n]$, and decoupled the square in the exponential using the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2 + az} dz = e^{a^2/2}. \quad (22)$$

Consequently the replica spins S^α (without site label) are independent of each other and feel a Gaussian random field (the same for all replicas) with zero mean and variance given by

$$\Delta^2 \equiv \beta^2 (J^2 q + h_r^2). \quad (23)$$

We denote an average over the Gaussian random variable z in Eq. (21) by $[\cdots]_z$, i.e.

$$[f(z)]_z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} f(z) dz. \quad (24)$$

It is straightforward to evaluate averages over the S^α , since they are independent, so we will now express averages over the real spins S_i in terms of S^α averages.

One can show, see e.g. Ref. 19, that each separate thermal average corresponds to a distinct replica, so, for example,

$$[\langle S_i S_j \rangle \langle S_k \rangle \langle S_l \rangle]_{\text{av}} = \langle S_i^\alpha S_j^\alpha S_k^\beta S_l^\gamma \rangle \quad (25)$$

for α, β and γ all different. To evaluate averages of the form in the RHS of Eq. (25) we add fictitious fields $\Delta_{\alpha\beta}$ which couple the replicas¹⁹, so Eq. (18) becomes

$$\begin{aligned} [Z^n]_{\text{av}} &= \text{Tr} \exp \left(\frac{(\beta J)^2}{2N} \sum_{\langle i,j \rangle} \sum_{\alpha,\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta \right. \\ &\quad \left. + \frac{h_r^2}{2} \sum_i \sum_{\alpha,\beta} S_i^\alpha S_i^\beta + \sum_{(\alpha\beta)} \Delta_{\alpha\beta} \sum_i S_i^\alpha S_i^\beta \right). \end{aligned} \quad (26)$$

Taking derivatives with respect to $\Delta_{\alpha\beta}$, one has, for $n \rightarrow 0$,

$$\sum_i \langle S_i^\alpha S_i^\beta \rangle = \frac{\partial}{\partial \Delta_{\alpha\beta}} [Z^n]_{\text{av}}, \quad (27a)$$

$$\sum_{i,j} \langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta \rangle = \frac{\partial^2}{\partial \Delta_{\alpha\beta} \partial \Delta_{\gamma\delta}} [Z^n]_{\text{av}}. \quad (27b)$$

Now setting the $\Delta_{\alpha\beta}$ to zero we get, from Eq. (26), in the $n \rightarrow 0$ limit,

$$q \equiv \frac{1}{N} [\langle S_i \rangle^2]_{\text{av}} = \frac{1}{N} \sum_i \langle S_i^\alpha S_i^\beta \rangle = [\langle S^\alpha S^\beta \rangle]_z, \quad (28)$$

for $\alpha \neq \beta$. We emphasize that, in the final average $[\cdots]_z$, the inner brackets refer to averaging over the spins in a fixed value of the random field z in Eq. (21), and the outer brackets, $[\cdots]_z$, refer to averaging over z according to Eq. (24). Equation (28) leads to the well-known self-consistent expression^{1,19} for the spin glass order parameter q :

$$\begin{aligned} q &= [\langle S^\alpha S^\beta \rangle]_z, \\ &= [\tanh^2[\beta (J^2 q + h_r^2)^{1/2} z]]_z, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \tanh^2[\beta (J^2 q + h_r^2)^{1/2} z] dz. \end{aligned} \quad (29)$$

It will be useful to express the average in Eq. (27a) in a different way. Including the fictitious fields $\Delta_{\alpha\beta}$ in the derivation which led from Eq. (18) to Eqs. (19) and (20) one finds an extra term, $\sum_{(\alpha\beta)} \Delta_{\alpha\beta} S^\alpha S^\beta$, in $L[q_{\alpha\beta}]$. Defining new integration variables by¹⁹

$$q_{\alpha\beta} + (\beta J)^{-2} \Delta_{\alpha\beta} \rightarrow q_{\alpha\beta}, \quad (30)$$

then $\Delta_{\alpha\beta}$ no longer appears in L , only in the quadratic term in Eq. (19). Using Eqs. (27), one then gets

$$q = \frac{1}{N} \sum_i \langle S_i^\alpha S_i^\beta \rangle = \langle q_{\alpha\beta} \rangle, \quad (31a)$$

$$\frac{1}{N} \sum_{i,j} \langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta \rangle = N \langle q_{\alpha\beta} q_{\gamma\delta} \rangle - (\beta J)^{-2} \delta_{(\alpha\beta),(\gamma\delta)}. \quad (31b)$$

Hence the spin glass susceptibility, defined in Eq. (11), is given by^{19,20}

$$\chi_{SG} = N \left(\langle \delta q_{\alpha\beta}^2 \rangle - 2 \langle \delta q_{\alpha\beta} \delta q_{\alpha\gamma} \rangle + \langle \delta q_{\alpha\beta} \delta q_{\gamma\delta} \rangle \right) - (\beta J)^{-2}, \quad (32)$$

where all replicas are different, and $\delta q_{\alpha\beta}$ is defined by

$$q_{\alpha\beta} = q + \delta q_{\alpha\beta}. \quad (33)$$

We now expand Eq. (19) about the saddle point to quadratic order in the $\delta q_{\alpha\beta}$. The result is that the exponential in Eq. (19) becomes

$$\exp \left(-N f(q) - N \frac{(\beta J)^2}{2} \sum_{(\alpha\beta),(\gamma\delta)} A_{(\alpha\beta),(\gamma\delta)} \delta q_{\alpha\beta} \delta q_{\gamma\delta} \right), \quad (34)$$

where $f(q)$ is the value of the exponent at the saddle point. To obtain the elements of the $\frac{1}{2}n(n-1)$ by $\frac{1}{2}n(n-1)$ matrix A we take the log of Eq. (19) and write the coefficients in the expansion of $\ln \text{Tr } e^L$ in powers of

the $\delta q_{\alpha\beta}$ in terms of spin averages, evaluated by the decoupling in Eq. (21). The result is

$$A_{(\alpha\beta),(\gamma\delta)} = \delta_{(\alpha\beta)(\gamma\delta)} - (\beta J)^2 \{ [\langle S^\alpha S^\beta S^\gamma S^\delta \rangle]_z - [\langle S^\alpha S^\beta \rangle]_z [\langle S^\gamma S^\delta \rangle]_z \}. \quad (35)$$

Equation (34) is the weight function used for averaging over the $\delta q_{\alpha\beta}$ in Eq. (32). Performing these Gaussian integrals gives

$$\chi_{SG} = \frac{1}{(\beta J)^2} \left[G_{(\alpha\beta),(\alpha\beta)} - 2G_{(\alpha\beta),(\alpha\gamma)} + G_{(\alpha\beta),(\gamma\delta)} - 1 \right], \quad (36)$$

where G is the matrix inverse of A , i.e.

$$G A = I \quad (37)$$

where I is the identity matrix. Defining

$$G_{(\alpha\beta),(\alpha\beta)} = G_1, \quad (38a)$$

$$G_{(\alpha\beta),(\alpha\gamma)} = G_2, \quad (38b)$$

$$G_{(\alpha\beta),(\gamma\delta)} = G_3, \quad (38c)$$

we have

$$\chi_{SG} = \frac{1}{(\beta J)^2} (G_r - 1), \quad (39)$$

where

$$G_r = G_1 - 2G_2 + G_3 \quad (40)$$

is called the “replicon propagator”²¹.

The matrix inverse of A is evaluated in Appendix A 4. According to Eq. (A22) we can express Eq. (39) as

$$\chi_{SG} = \frac{1}{(\beta J)^2} \left(\frac{1}{\lambda_3} - 1 \right), \quad (41)$$

where

$$\lambda_3 = P - 2Q + R, \quad (42)$$

and the quantities P, Q and R are defined in Eq. (A2). The eigenvalues of A are evaluated in Appendices A 1–A 3, and it turns out that λ_3 is an eigenvalue of A , see Eq. (A17). We evaluate the relevant spin averages needed to determine λ_3 in in Appendix C, and Eq. (C22) gives

$$\lambda_3 = 1 - (\beta J)^2 \chi_{SG}^0, \quad (43)$$

or equivalently, from Eq. (41),

$$\chi_{SG} = \frac{\chi_{SG}^0}{1 - (\beta J)^2 \chi_{SG}^0}, \quad (44)$$

where χ_{SG}^0 is a single-site spin glass susceptibility, given for the Ising case by

$$\begin{aligned} \chi_{SG}^0 &= \left[\left(\langle SS \rangle - \langle S \rangle \langle S \rangle \right)^2 \right]_z, \\ &= \left[\left(1 - \langle S \rangle^2 \right)^2 \right]_z, \\ &= \left[\left(1 - \tanh^2[\beta (J^2 q + h_r^2)^{1/2} z] \right)^2 \right]_z, \\ &= 1 - 2q + r, \end{aligned} \quad (45)$$

where q is given by Eq. (29) and r is given by

$$r = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \tanh^4[\beta (J^2 q + h_r^2)^{1/2} z] dz. \quad (46)$$

Hence, according to the RS ansatz, χ_{SG} is predicted to diverge where

$$(\beta J)^2 \chi_{SG}^0 = 1, \quad (47)$$

which describes the location of the AT line. In particular, for small fields the AT line is given by

$$h_r^2 = \frac{4}{3} \left(\frac{T_c - T}{T_c} \right)^3 \quad (m = 1), \quad (48)$$

see Eq. (C34). In fact, χ_{SG} turns out to be negative below this line since λ_3 is negative in this region, see Eq. (C33). These results were first found by AT. At low temperatures we get

$$\frac{h_r(T \rightarrow 0)}{J} = \sqrt{\frac{8}{9\pi}} \frac{J}{T} \quad (m = 1), \quad (49)$$

see Eq. (C37), in agreement with Bray⁷. A plot of the AT line for $m = 1$, obtained numerically, is shown in Fig. 1.

Although the derivation of Eq. (44) is rather involved, we note that the final answer is quite simple and has a familiar mean field form, i.e. a response function χ is equal to $\chi_0/(1 - K\chi_0)$ where χ_0 is the non-interacting response function, and K ($= (\beta J)^2$ here), is a coupling constant. In the next section, we will see that χ_{SG} has precisely the same mean field form for the vector ($m > 1$) case.

III. THE SPIN GLASS SUSCEPTIBILITY FOR VECTOR SPIN GLASSES

Here we consider a vector spin glass in which the Ising spins are replaced by vector spins with m components. The fluctuations in zero field were first considered by AJKT and Ref. 22 and our approach follows closely that of the latter reference. However, we shall see that there are some differences between our results and those of AJKT and Ref. 22. The derivation follows the lines of that for the Ising case in the previous section, but with

the burden of additional indices for the spin components. Hence we will not go through the details but just indicate the main steps and the results.

To avoid confusion in notation, we will use the Greek letters $\alpha, \beta, \gamma, \delta, \epsilon$ for replicas and μ, ν, κ, σ for spin indices. The auxiliary variables q will now involve 4 indices $(\alpha\beta), \mu\nu$, in which the order of the replica pair $(\alpha\beta)$ is unimportant, i.e. $(\beta\alpha)$ is the same as $(\alpha\beta)$, but the order of the spin indices does matter because $S_\alpha^\mu S_\beta^\nu$ is not the

same as $S_\alpha^\nu S_\beta^\mu$. Another new feature which appears when we deal with vector spins is the appearance of terms with both replicas equal, $(\alpha\alpha)$. These do not appear for the Ising case because $(S_\alpha)^2$ is equal to 1, a constant. However, $(S_\alpha^\mu)^2$ is not a constant for $m > 1$ and so we now need to include $(\alpha\alpha)$ terms in the analysis, though they will not enter the final result for χ_{SG} .

The analogues of Eqs. (19) and (20) are

$$[Z^n]_{\text{av}} = \int_{-\infty}^{\infty} \left(\prod_{(\alpha\beta), \mu, \nu} \left(\frac{N}{2\pi} \right)^{1/2} (\beta J) dq_{\alpha\beta}^{\mu\nu} \right) \left(\prod_{\alpha, \mu, \nu} \left(\frac{N}{2\pi} \right)^{1/2} (\beta J) dq_{\alpha\alpha}^{\mu\nu} \right) \times \exp \left(-N \frac{(\beta J)^2}{2} \left\{ \sum_{(\alpha\beta), \mu, \nu} (q_{\alpha\beta}^{\mu\nu})^2 + \sum_{\alpha, \mu, \nu} (q_{\alpha\alpha}^{\mu\nu})^2 \right\} \right) \left(\text{Tr} \exp L[q_{\alpha\beta}^{\mu\nu}, q_{\alpha\alpha}^{\mu\nu}] \right)^N, \quad (50)$$

where $L[q_{\alpha\beta}^{\mu\nu}, q_{\alpha\alpha}^{\mu\nu}]$ is given by

$$L[q_{\alpha\beta}^{\mu\nu}, q_{\alpha\alpha}^{\mu\nu}] = \beta^2 \sum_{(\alpha\beta), \mu, \nu} \left(J^2 q_{\alpha\beta}^{\mu\nu} + h_r^2 \delta_{\mu\nu} \right) S_\mu^\alpha S_\nu^\beta + \frac{(\beta J)^2}{\sqrt{2}} \sum_{\alpha, \mu, \nu} q_{\alpha\alpha}^{\mu\nu} S_\mu^\alpha S_\nu^\alpha, \quad (51)$$

where we ignored a term $\frac{1}{2}(\beta h_r)^2 \sum_{\mu, \alpha} (S_\mu^\alpha)^2$ since it is a constant.

We take the replica symmetric (RS) saddle point, where

$$q_{\alpha\beta}^{\mu\nu} = q \delta_{\mu\nu}, \quad q_{\alpha\alpha}^{\mu\nu} = x \delta_{\mu\nu}. \quad (52)$$

We then have, ignoring overall constant factors,

$$e^L \propto \text{Tr} \exp \left(\beta^2 \sum_{(\alpha\beta), \mu, \nu} (J^2 q + h_r^2) S_\mu^\alpha S_\nu^\beta \right) = \text{Tr} \exp \left(\frac{\beta^2}{2} (J^2 q + h_r^2) \sum_{\mu} \left(\left\{ \sum_{\alpha} S_\mu^\alpha \right\}^2 \right) - nm \right) \propto \int_{-\infty}^{\infty} \left(\prod_{\mu} \frac{dz_{\mu}}{\sqrt{2\pi}} \right) e^{-\sum_{\mu} z_{\mu}^2/2} \prod_{\alpha=1}^n \left[\text{Tr} e^{\beta(J^2 q + h_r^2)^{1/2} \sum_{\mu} z_{\mu} S_{\mu}^{\alpha}} \right], \quad (53)$$

where, to get the last line, we decoupled the square in the exponent using Eq. (22). As for the Ising case, we denote an average over the Gaussian random variables z_{μ}

by $[\dots]_z$.

Proceeding as in Sec. II, the spin glass susceptibility, defined in Eq. (16), is given by

$$\chi_{SG} = \frac{N}{m} \left(\sum_{\mu, \nu} \langle \delta q_{\alpha\beta}^{\mu\mu} \delta q_{\alpha\beta}^{\nu\nu} \rangle - 2 \langle \delta q_{\alpha\beta}^{\mu\mu} \delta q_{\alpha\gamma}^{\nu\nu} \rangle + \langle \delta q_{\alpha\beta}^{\mu\mu} \delta q_{\gamma\delta}^{\nu\nu} \rangle \right) - (\beta J)^{-2}, \quad (54)$$

(with α, β, γ and δ all different) where the averages over the δq are with respect to the following Gaussian weight (analogous to that in Eq. (34) for the Ising case),

$$\exp \left(-N \frac{(\beta J)^2}{2} \left\{ \sum_{(\alpha\beta), (\gamma\delta)} Z_{(\alpha\beta), (\gamma\delta)}^{\mu\nu, \kappa\sigma} \delta q_{\alpha\beta}^{\mu\nu} \delta q_{\gamma\delta}^{\kappa\sigma} + \sum_{(\alpha\alpha), (\gamma\delta)} Z_{(\alpha\alpha), (\gamma\delta)}^{\mu\nu, \kappa\sigma} \frac{\delta q_{\alpha\alpha}^{\mu\nu}}{\sqrt{2}} \delta q_{\gamma\delta}^{\kappa\sigma} + \sum_{\alpha, \gamma} Z_{(\alpha\alpha), (\gamma\gamma)}^{\mu\nu, \kappa\sigma} \frac{\delta q_{\alpha\alpha}^{\mu\nu}}{\sqrt{2}} \frac{\delta q_{\gamma\gamma}^{\kappa\sigma}}{\sqrt{2}} \right\} \right), \quad (55)$$

and

$$Z_{(\alpha\beta),(\gamma\delta)}^{\mu\nu,\kappa\sigma} = \delta_{(\alpha\beta)(\gamma\delta)} \delta_{\mu\kappa} \delta_{\nu\sigma} - (\beta J)^2 \left\{ [\langle S_\mu^\alpha S_\nu^\beta S_\kappa^\gamma S_\sigma^\delta \rangle]_z - [\langle S_\mu^\alpha S_\nu^\beta \rangle] [\langle S_\kappa^\gamma S_\sigma^\delta \rangle]_z \right\}. \quad (56)$$

Note that the annoying factors of $1/\sqrt{2}$ and $1/2$ in Eq. (55) can be removed simply by incorporating a factor of $1/\sqrt{2}$ into the $q_{\alpha\alpha}^{\mu\nu}$. Doing the averages in Eq. (54) using the Gaussian weight in Eq. (55) gives

$$\chi_{SG} = \frac{1}{(\beta J)^2} \left\{ \frac{1}{m} \sum_{\mu,\nu} \left[G_{(\alpha\beta),(\alpha\beta)}^{\mu\mu\nu\nu} - 2G_{(\alpha\beta),(\alpha\gamma)}^{\mu\mu\nu\nu} + G_{(\alpha\beta),(\gamma\delta)}^{\mu\mu\nu\nu} \right] - 1 \right\}, \quad (57)$$

where $G = Z^{-1}$. Using the definitions in Eqs. (B32), we have

$$\chi_{SG} = \frac{1}{(\beta J)^2} (G_r - 1) \quad (58)$$

where the “replicon” propagator is given by

$$G_r = G_{1L} + (m-1)G_{1T} - 2[G_{2L} + (m-1)G_{2T}] + G_{3L} + (m-1)G_{3T}. \quad (59)$$

The matrix inverse of Z is evaluated in Appendix B 5. According to Eq. (B35), we can express Eq. (58) as

$$\chi_{SG} = \frac{1}{(\beta J)^2} \left(\frac{1}{\lambda_{3S}} - 1 \right), \quad (60)$$

where

$$\lambda_{3S} = P_L + (m-1)P_T - 2[Q_L + (m-1)Q_T] + R_L + (m-1)R_T. \quad (61)$$

We determine the eigenvalues of Z in Appendices B 1–B 3, and show that λ_{3S} is an eigenvalue,

From Eq. (C22), we see that Eq. (58) can be written in the same form as for the Ising case, Eq. (44), where, for the case of general m , the single-site spin glass susceptibility χ_{SG}^0 is evaluated in Appendix C, and given by Eq. (C24).

The AT line is where $(\beta J)^2 \chi_{SG}^0 = 1$. Near T_c this is given by

$$\left(\frac{h_r}{J} \right)^2 = \frac{4}{m+2} t^3, \quad (62)$$

see Eq. (C34). The same expression was obtained by Gabay and Toulouse⁴ but for a uniform field, in which case it refers to a crossover rather than a sharp transition. Note that $h_r = 0$ for $m = \infty$, as expected since AJKT showed that the replica symmetric solution is stable in this limit. In the opposite limit, $T \rightarrow 0$, we find that the value of the AT field is finite for $m > 2$,

$$\frac{h_r(T=0)}{J} = \frac{1}{\sqrt{m-2}} \quad (m > 2), \quad (63)$$

see Eq. (C36), while $h_r(T \rightarrow 0)$ diverges for $m \leq 2$. The location of the AT line, obtained numerically, is plotted in Fig. 1 for several values of m .

Below the AT line, χ_{SG} is predicted to be negative, see Eq. (C33), which is impossible and shows that the RS solution (which we have assumed) is wrong in this region.

For $h_r = 0$, Eq. (C33) gives $\lambda_{3S} = -4t^2/(m+2)$, which disagrees with the unstable eigenvalue $-8t^2/(m+2)^2$ given by AJKT and Ref. 22. However, we note that the replicon propagator in Eq. (59) corresponds precisely to Eq. (3.5) of Ref. 23, and Eq. (62) also appears in the paper by Gabay and Toulouse⁴, so we are confident that Eq. (C33) is correct. Note too that we obtained the spin glass susceptibility, the divergence of which indicates the AT line, *directly* from the inverse of the matrix Z , the calculation of which is fairly simple, see Appendix B 5. The extra information that χ_{SG} is related to an eigenvalue, λ_{3S} , is not strictly needed to locate the AT line.

IV. NUMERICAL RESULTS

We have determined the location of the AT line numerically for $m = 1, 3$ and 10. For a given T and assumed value of h_r we solve for q self-consistently from Eq. (C18) and substitute into Eq. (C24) which gives λ_{3S} from Eq. (C22). The value of h_r is then adjusted until $\lambda_{3S} = 0$. The results are shown by the solid lines in Fig. 1. Also shown, by the dashed lines, is the approximate form in Eq. (62) which is valid close to the zero field transition temperature. For $m = 3$ this approximation actually works well down to rather low temperatures.

If the spins are normalized to have length 1 rather than $m^{1/2}$ one divides the horizontal scale in Fig. 1 by m and the vertical scale by $1/m^{1/2}$, so the zero field transition temperature would be $T_c = J/m$ and the zero temperature limit of the AT field would be $h_r = J/\sqrt{m(m-2)}$, for $m > 2$ (compare with Eq. (63)).

We have also checked these results by Monte Carlo simulations for the Heisenberg case, $m = 3$. The method has been discussed elsewhere^{16,18}, so here we just give a few salient features. We use three types of moves: heatbath, overrelaxation, and parallel tempering^{24,25}. We perform one heatbath sweep and one parallel tempering sweep for every ten overrelaxation sweeps. The parameters of the simulations are given in Table I. In calculating the spin glass susceptibility in Eq. (16), each thermal average is run in a separate copy of the system to avoid bias. Hence we simulate four copies at each temperature.

When the quenched random disorder variables are Gaussian, as here, the following identity is easily shown to hold by integrating by parts the expression for the average energy U with respect to the disorder variables^{16,26},

$$-\frac{U}{m} \equiv \frac{[\langle \mathcal{H} \rangle]_{\text{av}}}{m} = \frac{J^2}{2T} (q_s - q_l) + \frac{h_r^2}{T} (1 - \bar{q}), \quad (64)$$

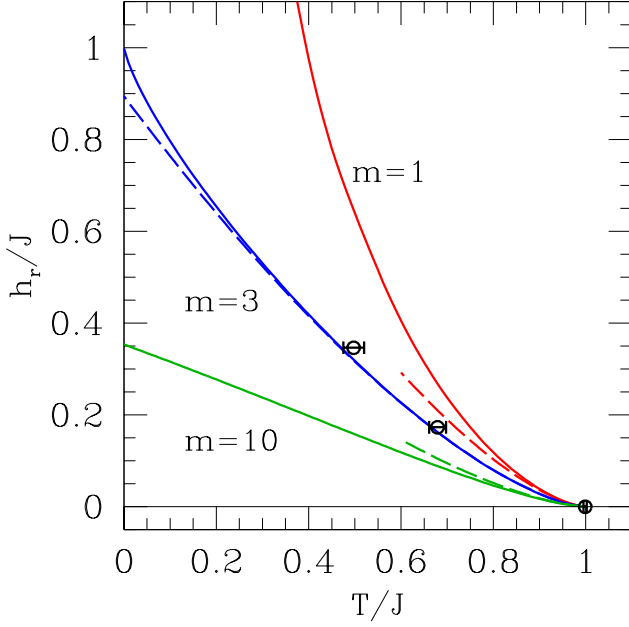


FIG. 1: The solid lines indicate the location of the AT line for $m = 1, 3$ and 10 , according to Eq. (71), and χ_{SG}^0 given by Eq. (C24). For $m \rightarrow \infty$ the AT line collapses on to the horizontal axis. The dashed lines are the approximate form given in Eq. (62), which is valid close to $T = T_c = J$. Note that this approximation works remarkably well for the Heisenberg case, $m = 3$, even down to quite low temperatures. Also shown are Monte Carlo results for the critical temperature for $h_r = 0, 0.173$ and 0.346 for $m = 3$.

where

$$q_s = \frac{1}{Nm} \sum_{i \neq j} [\langle (\mathbf{S}_i \cdot \mathbf{S}_j)^2 \rangle]_{\text{av}}, \quad (65)$$

$$q_l = \frac{1}{Nm} \sum_{i \neq j} [\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle^2]_{\text{av}}, \quad (66)$$

$$\bar{q} = \frac{1}{Nm} \sum_i [\langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_i \rangle]_{\text{av}}, \quad (67)$$

in which \bar{q} is the expectation value of the spin glass order parameter, and q_l is called the “link” overlap.

While Eq. (64) is true in equilibrium, is not true before equilibrium is reached, and, indeed, the two sides of the equation approach the equilibrium value from opposite directions^{16,26}. Hence we only accept the results of a simulation if Eq. (64) is satisfied with small error bars. (Note that this equation refers to an average over samples; the connection between the energy and the spin correlations is not true sample by sample.)

According to finite-size scaling the spin glass susceptibility in a finite, infinite-range system, should vary as^{27–30}

$$\chi_{SG} = N^{1/3} \tilde{X} \left(N^{1/3} (T - T_c(h_r)) \right), \quad (68)$$

TABLE I: Parameters of the simulations for different values of h_r . Here N_{samp} is the number of samples, N_{sweep} is the number of overrelaxation Monte Carlo sweeps, T_{min} and T_{max} are the lowest and highest temperatures simulated, and N_T is the number of temperatures.

h_r	N	N_{samp}	N_{sweep}	T_{min}	T_{max}	N_T
0	64	8000	256	0.30	1.50	40
0	128	8000	512	0.30	1.50	40
0	256	8000	1024	0.30	1.50	40
0	512	8000	2048	0.30	1.50	40
0	1024	2078	4096	0.30	1.50	40
0.173	64	8000	1024	0.30	1.50	40
0.173	128	8000	2048	0.30	1.50	40
0.173	256	8000	4096	0.30	1.50	40
0.173	512	4279	8192	0.30	1.50	40
0.173	1024	1494	16384	0.39	1.50	40
0.346	64	8000	1024	0.15	1.20	40
0.346	128	8000	2048	0.15	1.20	40
0.346	256	8000	4096	0.15	1.20	40
0.346	512	4293	8192	0.15	1.20	40
0.346	1024	3037	16384	0.15	1.20	40

so plots of $\chi_{SG}/N^{1/3}$ should intersect at the transition temperature $T_c(h_r)$. Data for $\chi_{SG}/N^{1/3}$ for $m = 3$ for random field values $h_r = 0, 0.173$ and 0.346 , are shown in Figs. 2, 3 and 4. The data does indeed intersect, indicating a transition, though the data for different sizes don’t intersect at exactly the same temperature which indicates the presence of corrections to finite-size scaling.

There are both singular and analytic corrections to scaling. In the mean field limit the leading correction to χ_{SG} is analytic³¹, in fact just a constant, so we replace Eq. (68) by

$$\chi_{SG} = N^{1/3} \tilde{X} \left(N^{1/3} (T - T_c(h_r)) \right) + c_0. \quad (69)$$

We compute the intersection temperature $T^*(N, 2N)$ between data for $\chi_{SG}/N^{1/3}$ for sizes N and $2N$. It is then easy to see from Eq. (69) that the $T^*(N, 2N)$ converge to the transition temperature like

$$T^*(N, 2N) - T_c(h_r) = \frac{A}{N^{2/3}}, \quad (70)$$

where the constant A is related to c_0 and $\tilde{X}'(0)$. We determine $T^*(N, 2N)$ by a bootstrap analysis and show the results both in Table II and in the insets to Figs. 2, 3 and 4. Fitting a straight line through $T^*(N, 2N)$ against $N^{-2/3}$ according to Eq. (70), gives estimates of T_c which shown both in Fig. 1 and Table II.

We see that, in zero field, the numerics accurately gives the exact value for T_c of 1, and for non-zero h_r , the numerics gives the correct answer to within about one sigma. Hence our analytical predictions for the AT line in Heisenberg spin glasses are well confirmed by simulations.

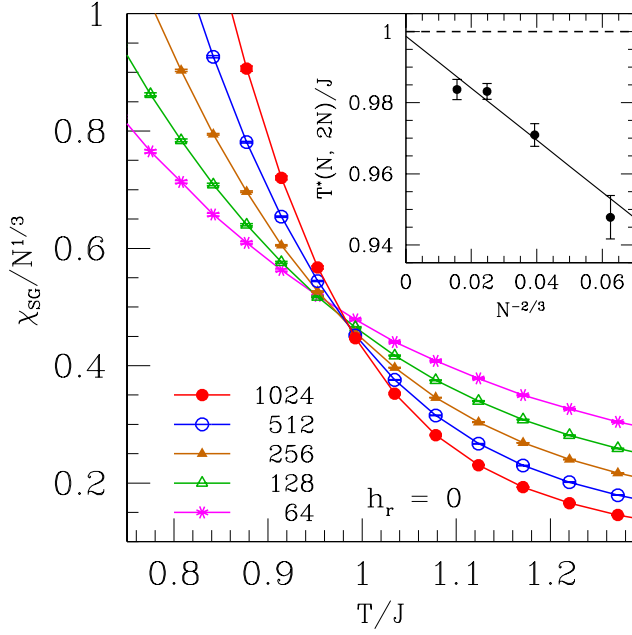


FIG. 2: Zero field Monte Carlo data for the spin glass susceptibility for the $m = 3$ (Heisenberg) infinite-range spin glass, divided by $N^{1/3}$, for different sizes. According to finite-size scaling, the data should intersect at the transition temperature T_c in the absence of corrections to scaling. Allowing for the leading corrections, the inset shows intersection temperatures $T^*(N, 2N)$ for sizes N and $2N$ and the extrapolation to $N = \infty$ according to Eq. (70). This leads to the estimate $T_c = 0.9987 \pm 0.0036$ (see Table II), which agrees well with the exact value of 1, shown as the dashed line in the inset.

TABLE II: Intersection temperatures $T^*(N, 2N)$, and extrapolated values of $T_c(h_r)$ determined from fits to Eq. (70). Also shown is the exact value for $T_c(h_r)$, obtained as described in the text.

h_r	N	$T^*(N, 2N)$	$T_c(h_r)$	$T_c(h_r)$ (exact)
0	64	0.9478(61)		
0	128	0.9709(32)		
0	256	0.9832(22)		
0	512	0.9837(29)		
0	∞		0.9987(36)	1
0.173	64	0.633(13)		
0.173	128	0.634(13)		
0.173	256	0.668(15)		
0.173	512	0.680(18)		
0.173	∞		0.679(19)	0.6652
0.346	64	0.473(19)		
0.346	128	0.447(29)		
0.346	256	0.485(19)		
0.346	512	0.498(21)		
0.346	∞		0.497(23)	0.4706

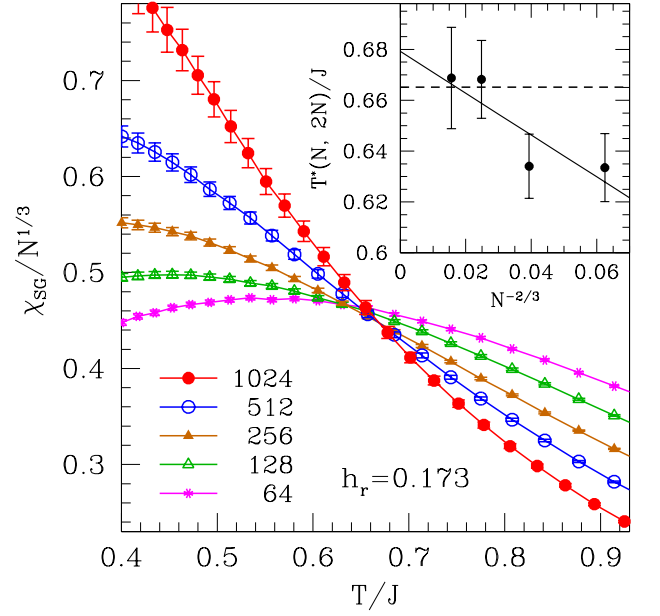


FIG. 3: Same as Fig. 2 but for random field strength $h_r = 0.173$. The final estimate of $T_c(h_r)$ is 0.685 ± 0.019 which is to be compared with the exact value of 0.6652, see Table II, which is shown as the dashed line in the inset.

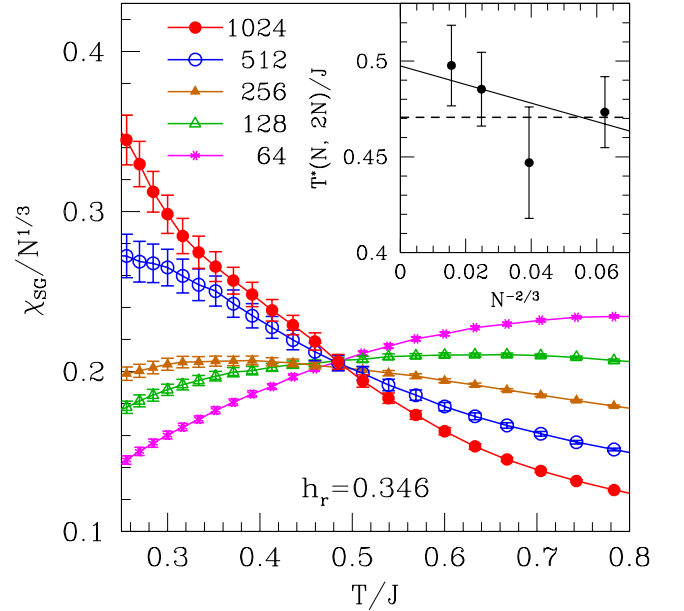


FIG. 4: Same as Fig. 2 but for random field strength $h_r = 0.346$. The final estimate of $T_c(h_r)$ is 0.497 ± 0.023 , to be compared with the exact value of 0.4706, see Table II, which is shown as the dashed line in the inset.

V. CONCLUSIONS

We have emphasized that the appropriate symmetry breaking field for a spin glass is a random field, and that, for a vector spin glass, the crucial ingredient is the random *direction* of the field. Incorporating a random field, there is a line of transitions (AT line) in vector spin glasses, just as there is in the Ising spin glass, a fact which does not seem to be widely recognized. The AT line is different from the Gabay-Toulouse⁴ (GT) line, since it is a transition to a phase with replica symmetry breaking but *no change in spin symmetry*.

The location of the AT line for vector spin glasses with Gaussian random fields is given by

$$\left(\frac{T}{J}\right)^2 = \chi_{SG}^0, \quad (71)$$

where χ_{SG}^0 is given by Eq. (C24). For the important case of the Heisenberg ($m = 3$) spin glass, the simpler expression for χ_{SG}^0 is given in Eq. (C26). We have plotted the AT line numerically for several values of m in Fig. 1, and confirmed these results numerically by simulations for the case of $m = 3$.

For the Ising case, we note that Bray and Moore³² have obtained Eq. (44) for the spin glass susceptibility without replicas, starting from the local mean-field equations of Thouless, Anderson and Palmer³³ (the TAP equations). It would be interesting to see if one could derive, along similar lines, a more straightforward, and non-replica, calculation of χ_{SG} for the vector spin case too.

Although it is not possible experimentally to apply a field which is random in direction to a vector spin glass, so the AT line seems to be experimentally inaccessible (except for the Ising case), one *can* detect the AT line for vector spin glasses in simulations. Whether or not at AT line exists in finite-range spin glasses, is a crucial difference between the replica symmetry breaking (RSB) picture, where it does occur, and the droplet picture, where it does not. It has been found possible to simulate Heisenberg spin glasses for significantly larger sizes^{16–18} than Ising spin glasses, so our results may give an additional avenue through which to investigate the nature of the spin glass state.

Acknowledgments

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Appendix A: Fluctuation analysis for Ising spin glasses

We follow AT in obtaining the eigenvalues and eigenvectors (and also the inverse, not calculated by AT) of the real symmetric matrix A of dimension $n(n-1)/2$, in which each row of column is labeled by a pair of distinct spin indices $(\alpha\beta)$, with elements given by (see Eq. (35))

$$A_{(\alpha\beta),(\gamma\delta)} = \delta_{(\alpha\beta)(\gamma\delta)} - (\beta J)^2 \{ [\langle S^\alpha S^\beta S^\gamma S^\delta \rangle]_z - [\langle S^\alpha S^\beta \rangle]_z [\langle S^\gamma S^\delta \rangle]_z \}. \quad (A1)$$

where the average $\langle \dots \rangle$ is over the spins for a fixed value of the Gaussian random field z in Eq. (21), and the average $[\dots]_z$ is over z according to Eq. (24).

Because the theory is invariant under permutation of the replicas, there are only three distinct values for the matrix elements:

$$A_{(\alpha\beta),(\alpha\beta)} = P, \quad (A2a)$$

$$A_{(\alpha\beta),(\alpha\gamma)} = Q, \quad (A2b)$$

$$A_{(\alpha\beta),(\gamma\delta)} = R, \quad (A2c)$$

in which α, β, γ and δ are all different. Recall that $(\alpha\beta)$ takes $n(n-1)/2$ distinct values, i.e. the pair $(\beta\alpha)$ is the same as the pair $(\alpha\beta)$.

1. First eigenvalue and eigenvector

If we go along *any* row or column, the number of times, P, Q and R appear is given by

$$n_P = 1, \quad (A3a)$$

$$n_Q = 2(n-2), \quad (A3b)$$

$$n_R = \frac{1}{2}(n-2)(n-3). \quad (A3c)$$

Since the sum of all elements in any row or column is the same for each row and column, it trivially follows that there is an eigenvector

$$\vec{e}_1 = (1, 1, \dots, 1), \quad (A4)$$

with eigenvalue equal to the sum of all the elements along a row or column,

$$\lambda_1 = P + 2(n-2)Q + \frac{1}{2}(n-2)(n-3)R. \quad (A5)$$

This eigenvalue has degeneracy 1.

2. Second eigenvalue and eigenvectors

We look for an eigenvector $\vec{e}_{2,\epsilon}$ with elements

$$e_{2,\epsilon}^{\alpha\beta} = \begin{cases} d & (\text{if } \alpha \text{ or } \beta = \epsilon), \\ e & (\text{otherwise}), \end{cases} \quad (A6)$$

for some ϵ . The $\vec{e}_{2,\epsilon}$ must be orthogonal to \vec{e}_1 in Eq. (A4), which means

$$\sum_{(\alpha\beta)} e_{2,\epsilon}^{(\alpha\beta)} = 0, \quad (\text{A7})$$

for each ϵ , and so

$$(n-2)e = -2d. \quad (\text{A8})$$

Naively there are n independent vectors since there are n choices for ϵ . However, these are not all independent since it is quite easy to show that

$$\sum_{\epsilon} \vec{e}_{2,\epsilon} = 0. \quad (\text{A9})$$

Hence there is one linear constraint among the n vectors defined by Eq. (A6) and so the number of linearly independent such vectors is $n-1$, i.e. the degeneracy is $n-1$. It is now straightforward to verify from Eqs. (A6) and (A8), that

$$A \vec{e}_{2,\epsilon} = \lambda_2 \vec{e}_{2,\epsilon}, \quad (\text{A10})$$

where λ_2 is the eigenvalue, given by

$$\lambda_2 = P + (n-4)Q - (n-3)R. \quad (\text{A11})$$

Note that $\lambda_2 = \lambda_1$ for $n \rightarrow 0$.

3. Third eigenvalue and eigenvectors

We look for an eigenvector $\vec{e}_{3,(\eta\sigma)}$ with elements

$$e_{3,(\eta\sigma)}^{(\alpha\beta)} = \begin{cases} f & \text{(if } (\alpha\beta) = (\eta\sigma)), \\ g & \text{(if one of } (\alpha\beta) \text{ is equal to one of } (\eta\sigma)), \\ h & \text{(if } (\alpha\beta) \neq (\eta\sigma)), \end{cases} \quad (\text{A12})$$

for some choice of η and σ (with $\sigma \neq \eta$). The vectors in Eq. (A12) must be orthogonal to \vec{e}_1 in Eq. (A4), and to the $\vec{e}_{2,\epsilon}$ in Eq. (A6) so

$$f = (2-n)g, \quad g = \frac{1}{2}(3-n)h. \quad (\text{A13})$$

One can show that summing over one of the indices labeling a vector, gives zero, i.e.

$$\sum_{\eta} \vec{e}_{3,(\eta\sigma)} = 0. \quad (\text{A14})$$

Equation (A14) gives n constraints, one for each value of σ . Hence the number of linearly independent eigenvectors of the third type is $n(n-1)/2$ (the number of values of the index $(\eta\sigma)$ less n , the number of linear constraints. Hence the degeneracy is $\frac{1}{2}n(n-3)$. One can also show that the sum over one of the replica component indices vanishes for each vector³⁴, i.e.

$$\sum_{\alpha} e_{3,(\eta\sigma)}^{(\alpha\beta)} = 0. \quad (\text{A15})$$

(Recall that the subscript indices $(\eta\sigma)$ indicate a particular vector, and the superscript indices $(\alpha\beta)$ denote a particular element of that vector.)

It is now straightforward to show that the vectors in Eq. (A12) are indeed eigenvectors, i.e.

$$A \vec{e}_{3,(\eta\sigma)} = \lambda_3 \vec{e}_{3,(\eta\sigma)}, \quad (\text{A16})$$

where λ_3 is the “replicon” eigenvalue,

$$\lambda_3 = P - 2Q + R. \quad (\text{A17})$$

The total number of eigenvectors, of type 1, 2 or 3, found so far is $1 + (n-1) + \frac{1}{2}n(n-3) = \frac{1}{2}n(n-1)$, which is the dimension of the matrix. Hence we have found *all* the eigenvalues and eigenvectors.

4. Matrix inverse

Consider the matrix G which is the inverse of A , i.e.

$$A G = I \quad (\text{A18})$$

where I is the identity matrix. We assume that G has the same structure as A and define, see Eq. (38),

$$G_{(\alpha\beta),(\alpha\beta)} = G_1, \quad (\text{A19a})$$

$$G_{(\alpha\beta),(\alpha\gamma)} = G_2, \quad (\text{A19b})$$

$$G_{(\alpha\beta),(\gamma\delta)} = G_3, \quad (\text{A19c})$$

Evaluating the $(\alpha\beta),(\alpha\beta)$, the $(\alpha\beta),(\alpha\gamma)$, and the $(\alpha\beta),(\gamma\delta)$ elements of Eq. (A18) gives respectively

$$P G_1 + 2(n-2)Q G_2 + \frac{1}{2}(n-2)(n-3)R G_3 = 1. \quad (\text{A20a})$$

$$Q G_1 + [P + (n-2)Q + (n-3)R] G_2 + [(n-3)Q + \frac{1}{2}(n-3)(n-4)R] G_3 = 0. \quad (\text{A20b})$$

$$R G_1 + [4Q + 2(n-4)R] G_2 + [P + 2(n-4)Q + \frac{1}{2}(n-4)(n-5)R] G_3 = 0. \quad (\text{A20c})$$

Taking $1 \times (\text{A20a}) - 2 \times (\text{A20b}) + 1 \times (\text{A20c})$ gives

$$(P - 2Q + R)(G_1 - 2G_2 + G_3) = 1, \quad (\text{A21})$$

so the “replicon propagator” is given by

$$G_r \equiv G_1 - 2G_2 + G_3 = \frac{1}{P - 2Q + R}. \quad (\text{A22})$$

The spin glass susceptibility is determined from G_r according to Eq. (39). Note that Eqs. (A21) and (39) determine χ_{SG} without needing to diagonalize the matrix A . However, since the diagonalization has been done by AT it is instructive to see that G_r is the inverse of the replicon eigenvalue in Eq. (A17), see also Appendix A.

If we accept that λ_3 is an eigenvalue then Eq. (A22) is obvious since the eigenvectors of A and G are the same, and the corresponding eigenvalues are the inverses of each other. Furthermore, since the inverse matrix G has the same structure as that of the original matrix A , the expressions for the eigenvalues of A in terms of the parameters P, Q and R , are the same as the expressions for the eigenvalues of G in terms of the corresponding parameters G_1, G_2 and G_3 .

Appendix B: Fluctuation analysis for vector spin glasses

We now have additional indices for the spin components, and to avoid confusion in notation, we will use Greek letters $\alpha, \beta, \gamma, \delta, \epsilon$ for replicas and μ, ν, κ, σ for spin indices. A row or column of the matrix will then involve 4 indices $(\alpha\beta), \mu\nu$, in which the order of the replica pair $(\alpha\beta)$ is unimportant, i.e. $(\beta\alpha)$ is the same as $(\alpha\beta)$, but the order of the spin indices does matter because $S_\alpha^\mu S_\beta^\nu$ is not the same as $S_\alpha^\nu S_\beta^\mu$.

Another new feature which appears when we deal with vector spins is the appearance of terms with both replicas equal, $(\alpha\alpha)$. These do not appear for the Ising case because $(S_\alpha)^2$ is equal to 1, a constant. However, $(S_\alpha^\mu)^2$ is not a constant for $m > 1$ and so we now need to include $(\alpha\alpha)$ terms in the analysis.

Hence we shall need to find the eigenvalues and eigenvectors of a matrix Z of size $\frac{1}{2}n(n+1)m^2$ whose elements are given by

$$Z_{(\alpha\beta),(\gamma\delta)}^{\mu\nu,\kappa\sigma} = \delta_{(\alpha\beta)(\gamma\delta)} \delta_{\mu\kappa} \delta_{\nu\sigma} - (\beta J)^2 \left\{ [\langle S_\mu^\alpha S_\nu^\beta S_\kappa^\gamma S_\sigma^\delta \rangle]_z - [\langle S_\mu^\alpha S_\nu^\beta \rangle] [\langle S_\kappa^\gamma S_\sigma^\delta \rangle]_z \right\}. \quad (\text{B1})$$

Ignoring for now the spin indices (which will be put back later) we consider the following matrix of dimension $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$,

$$Z = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \quad (\text{B2})$$

in which A is the matrix of dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ with rows and columns labeled by two distinct replicas $(\alpha\beta)$ defined in Eq. (A2), C is an $n \times n$ matrix with rows and columns labeled by a single replica $(\alpha\alpha)$, and B is a matrix with $\frac{1}{2}n(n-1)$ rows and n columns.

1. Decomposing into subspaces

We now discuss each of these matrices in turn.

- The elements of A are given by Eq. (A2).
- The elements of B are

$$B_{(\alpha\beta),(\alpha\alpha)} = S, \quad (\text{B3a})$$

$$B_{(\alpha\beta),(\gamma\gamma)} = T, \quad (\text{B3b})$$

in which α, β and γ are all different.

- The elements of C are

$$C_{(\alpha\alpha),(\alpha\alpha)} = U, \quad (\text{B4a})$$

$$C_{(\alpha\alpha),(\beta\beta)} = V, \quad (\text{B4b})$$

in which α and β are different.

Now we add the Cartesian spin indices. The result is that each element of the matrix Z in Eq. (B2) becomes an $m^2 \times m^2$ matrix with rows and columns labeled by a pair of spin component indices μ and ν , each of which runs over values from 1 to m .

A simplification is that the only non-zero elements are those where each Cartesian spin component occurs an even number of times (combining the row and column indices). Hence each $m^2 \times m^2$ matrix breaks up into different blocks. There is one $m \times m$ block, $(\mu\mu, \nu\nu)$ where $\mu = 1, \dots, n$, $\nu = 1, \dots, m$, and $m(m-1)/2$ blocks of size 2, $(\mu\nu, \mu\nu)$ and $(\mu\nu, \nu\mu)$ where μ and ν ($\neq \mu$) are fixed.

Consider, for example, one of the elements in A with value P , see Eq. (A2). This is now expanded into an $m^2 \times m^2$ matrix which is block diagonalized into

- (i) one $m \times m$ matrix, with rows and columns labeled by $\mu\mu$ ($\mu = 1, \dots, m$),

$$\begin{pmatrix} P_L & P_T & \dots & P_T \\ P_T & P_L & \dots & P_T \\ \vdots & \vdots & \ddots & \vdots \\ P_T & P_T & \dots & P_L \end{pmatrix}, \quad (\text{B5})$$

where the diagonal elements (to which we give the subscript L) are different from the off-diagonal elements (to which we give the subscript T), and

- (ii) $m(m-1)/2$ identical matrices of size 2×2 , with rows and columns labeled by $\mu\nu$ and $\nu\mu$ (for fixed μ and ν with $\mu \neq \nu$),

$$\begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix}, \quad (\text{B6})$$

in which we give the subscript “1” to the (equal) diagonal elements and the subscript “2” to the off-diagonal elements.

The eigenvalues of (B5) are

$$P_S = P_L + (m-1)P_T, \quad (\text{degeneracy } 1), \quad (\text{B7a})$$

$$P_A = P_L - P_T, \quad (\text{degeneracy } m-1), \quad (\text{B7b})$$

and those of (B6) are

$$P_+ = P_1 + P_2, \quad (\text{B8a})$$

$$P_- = P_1 - P_2, \quad (\text{B8b})$$

each of degeneracy 1.

The R, S, T, U and V elements of the replica matrix, in Eqs. (A2), (B3) and (B4), expand out into the same block structure in spin-component space.

However, we shall now show that things are somewhat different for the Q elements, which have replica structure $(\alpha\beta), (\alpha\gamma)$, i.e. *one* of the replicas is repeated. The order of the replica indices in a pair does not matter but, to keep track of which spin index goes with which replica

index, we should adopt some convention, e.g. put the lower replica index first. Consider then a situation with $\beta < \gamma$ and different values of α . The Q element involving these three replicas would then be labeled differently depending on the value of α relative to β and γ as follows:

$$(\alpha\beta), (\alpha\gamma) \quad (\alpha < \beta < \gamma), \quad (\text{B9a})$$

$$(\beta\alpha), (\alpha\gamma) \quad (\beta < \alpha < \gamma), \quad (\text{B9b})$$

$$(\beta\alpha), (\gamma\alpha) \quad (\beta < \gamma < \alpha), \quad (\text{B9c})$$

Hence the 2×2 matrix Q has the form

$$\begin{array}{c|cc} & \mu\nu & \nu\mu \\ \hline \mu\nu & Q_1 & Q_2 \\ \nu\mu & Q_2 & Q_1 \end{array} \quad (\text{B10})$$

for $\alpha < \beta < \gamma$ and $\beta < \gamma < \alpha$, while it is

$$\begin{array}{c|cc} & \mu\nu & \nu\mu \\ \hline \mu\nu & Q_2 & Q_1 \\ \nu\mu & Q_1 & Q_2 \end{array} \quad (\text{B11})$$

for $\beta < \alpha < \gamma$, i.e. Q_1 and Q_2 are interchanged in the latter case. This does not affect $Q_+ \equiv Q_1 + Q_2$ but it changes the sign of Q_- when the repeated replica index (α here) lies in between the other two (β and γ here).

Our goal is to diagonalize the matrix Z given by Eq. (B2), in which A, B and C , are matrices in replica space given by Eqs. (A2), (B3) and (B4), and each element in these matrices is itself an $m^2 \times m^2$ replica in spin-component space which block diagonalizes as discussed above. Symbolically we want to find the eigenvalues λ and eigenvectors (\vec{e}, \vec{f}) of

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{f} \end{pmatrix} = \lambda \begin{pmatrix} \vec{e} \\ \vec{f} \end{pmatrix}, \quad (\text{B12})$$

where the vector \vec{e} is of dimension $\frac{1}{2}n(n-1)m^2$ and \vec{f} is of dimension nm^2 .

Because the block structure in spin-component space is the same for all elements of Z in Eq. (B2) (except for the some aspects of the “−” sector), we can diagonalize separately the spin-component and replica sectors. Hence the eigenvalue equation, Eq. (B12), breaks up into 4 simpler sets equations, one set for each distinct spin-component sector:

- 1 set of equations of the type

$$\begin{pmatrix} A_S & B_S \\ B_S^T & C_S \end{pmatrix} \begin{pmatrix} \vec{e}_S \\ \vec{f}_S \end{pmatrix} = \lambda_S \begin{pmatrix} \vec{e}_S \\ \vec{f}_S \end{pmatrix}, \quad (\text{B13})$$

- $m-1$ identical sets of equations of the type

$$\begin{pmatrix} A_A & B_A \\ B_A^T & C_A \end{pmatrix} \begin{pmatrix} \vec{e}_A \\ \vec{f}_A \end{pmatrix} = \lambda_A \begin{pmatrix} \vec{e}_A \\ \vec{f}_A \end{pmatrix}, \quad (\text{B14})$$

- $m(m-1)/2$ identical sets of equations of the type

$$\begin{pmatrix} A_+ & B_+ \\ B_+^T & C_+ \end{pmatrix} \begin{pmatrix} \vec{e}_+ \\ \vec{f}_+ \end{pmatrix} = \lambda_+ \begin{pmatrix} \vec{e}_+ \\ \vec{f}_+ \end{pmatrix}, \quad (\text{B15})$$

- and $m(m-1)/2$ identical sets of equations of the type

$$\begin{pmatrix} A_- & B_- \\ B_-^T & C_- \end{pmatrix} \begin{pmatrix} \vec{e}_- \\ \vec{f}_- \end{pmatrix} = \lambda_- \begin{pmatrix} \vec{e}_- \\ \vec{f}_- \end{pmatrix}. \quad (\text{B16})$$

The matrices in Eqs. (B13)–(B16) are of dimension $n(n+1)/2$, while the vectors \vec{e} are of length $n(n-1)/2$ and the vectors \vec{f} are of length n .

Each of the sets of equations, (B13)–(B16) has the same structure, which is a little more complicated than diagonalizing the matrix A , described in the first part of this report, because the off-diagonal piece B couples the elements of A to the $n \times n$ block C , in which each row or column index has two equal replicas. However, we shall see that the square blocks A and C decouple in two cases: (i) the “−” sector, and (ii) the replicon eigenvectors in the S, A , and “+” sectors.

We shall first discuss the S, A and “+” spin-component sectors together, and then do the “−” sector which has to be treated separately.

2. The S, A and “+” Spin-Component Sectors

The matrices for these sectors are all the same provided one replaces the elements of the replica matrix Z in Eq. (B2) by the appropriate eigenvalue of the spin-component sector, see Eqs. (B7)–(B8) for the case of P .

We first discuss the replicon subspace.

a. Replicon Modes

Let us see if the replicon eigenvector, computed for the Ising case in Sec. A 3, satisfies Eq. (B2) with $\vec{f} = 0$, i.e.

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \vec{e}_3 \\ 0 \end{pmatrix} = \lambda_3 \begin{pmatrix} \vec{e}_3 \\ 0 \end{pmatrix}, \quad (\text{B17})$$

which requires

$$\sum_{(\alpha\beta)} B_{(\alpha\beta),(\gamma\gamma)} e_3^{(\alpha\beta)} = 0, \quad (\text{B18})$$

for each γ . From Eq. (B3) we have

$$\sum_{(\alpha\beta)} B_{(\alpha\beta),(\gamma\gamma)} e_3^{(\alpha\beta)} = 2S \sum_{\beta} e_3^{(\gamma\beta)} + T \sum_{\alpha \neq \gamma, \beta \neq \gamma} e_3^{(\alpha\beta)}. \quad (\text{B19})$$

The first term vanishes because of Eq. (A15). Again using Eq. (A15), the sum in the second term can be written as $-\sum_{\beta \neq \gamma} e_3^{(\gamma\beta)}$, which again vanishes by Eq. (A15). Hence Eq. (B18) is satisfied.

As a result, we don't need to do any more work to get the eigenvalues in the replicon sector for the vector spin glass. We just use Eq. (A17) for each of the S , A and “+” spin-component sectors in Eqs. (B13)–(B15), i.e.

$$\begin{aligned}\lambda_{3S} &= P_S - 2Q_S + R_S \\ &= [P_L + (m-1)P_T] - 2[Q_L + (m-1)Q_T] + \\ &\quad [R_L + (m-1)R_T],\end{aligned}\quad (\text{B20a})$$

$$\begin{aligned}\lambda_{3A} &= P_A - 2Q_A + R_A \\ &= (P_L - P_T) - 2(Q_L - Q_T) + (R_L - R_T)\end{aligned}\quad (\text{B20b})$$

$$\begin{aligned}\lambda_{3+} &= P_+ - 2Q_+ + R_+ \\ &= (P_1 + P_2) - 2(Q_1 + Q_2) + (R_1 + R_2).\end{aligned}\quad (\text{B20c})$$

As discussed above, the spin-component degeneracies of the S , A , and “+” subspaces are 1, $(m-1)$, and $\frac{1}{2}m(m-1)$ respectively. To get the overall degeneracies of the eigenvalues in Eq. (B20) one has to multiply these factors by the degeneracy in replica space, $\frac{1}{2}n(n-3)$.

b. “ λ_1 ” Modes

Referring to Eq. (B13) we look for a solution where all the elements of \vec{e}_{S1} are equal to a , say and all the elements of \vec{f}_{S1} are equal to b . This gives the coupled equations

$$\begin{pmatrix} \alpha_{1S} & \beta_{1S} \\ \frac{1}{2}(n-1)\beta_{1S} & \gamma_{1S} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_{1S} \begin{pmatrix} a \\ b \end{pmatrix}\quad (\text{B21})$$

where

$$\alpha_{1S} = [P_L + (m-1)P_T] + 2(n-2)[Q_L + (m-1)Q_T] + \frac{1}{2}(n-2)(n-3)[R_L + (m-1)R_T],\quad (\text{B22a})$$

$$\beta_{1S} = 2[S_L + (m-1)S_T] + (n-2)[T_L + (m-1)T_T],\quad (\text{B22b})$$

$$\gamma_{1S} = [U_L + (m-1)U_T] + (n-1)[V_L + (m-1)V_T].\quad (\text{B22c})$$

The eigenvalues are given by the solutions of the resulting quadratic equation

$$\lambda_{1a,S} = \frac{1}{2} \left[\alpha_{1S} + \gamma_{1S} + \sqrt{(\alpha_{1S} - \gamma_{1S})^2 + 2(n-1)\beta_{1S}^2} \right],\quad (\text{B23a})$$

$$\lambda_{1b,S} = \frac{1}{2} \left[\alpha_{1S} + \gamma_{1S} - \sqrt{(\alpha_{1S} - \gamma_{1S})^2 + 2(n-1)\beta_{1S}^2} \right].\quad (\text{B23b})$$

This calculation simply repeats for the A and “+” sectors with the appropriate substitutions for α, β and γ . The spin-component degeneracies for the S , A and “+” sectors are 1, $m-1$ and $m(m-1)/2$ respectively. These have to be multiplied by the degeneracy from the replica sector, which is just 1 in this case.

c. “ λ_2 ” Modes

We follow the procedure of Sec. A 2, by looking for an eigenvector in which one replica, ϵ say, is distinct from the others. Referring to Eq. (B14), we set $\vec{e}_{S2}^{\alpha\beta}$ equal to d if α or β are equal to ϵ , and equal to e otherwise. Orthogonality to the λ_1 eigenvector requires $(n-2)e = -2d$, see Eq. (A8). Similarly we set \vec{f}_{S2}^α equal to f if $\alpha = \epsilon$ and equal to g otherwise. Orthogonality to the λ_1 eigenvector requires $(n-1)g = -f$.

Substituting into Eq. (B14) then gives the coupled equations

$$\begin{pmatrix} \alpha_{2S} & \beta_{2S} \\ (n-2)\beta_{2S} & \gamma_{2S} \end{pmatrix} \begin{pmatrix} d \\ g \end{pmatrix} = \lambda_{1S} \begin{pmatrix} d \\ g \end{pmatrix}\quad (\text{B24})$$

where

$$\alpha_{2S} = [P_L + (m-1)P_T] + (n-4)[Q_L + (m-1)Q_T] - (n-3)[R_L + (m-1)R_T],\quad (\text{B25a})$$

$$\beta_{2S} = [S_L + (m-1)S_T] - [T_L + (m-1)T_T],\quad (\text{B25b})$$

$$\gamma_{2S} = [U_L + (m-1)U_T] - [V_L + (m-1)V_T].\quad (\text{B25c})$$

The eigenvalues are given by the solutions of the resulting quadratic equation

$$\lambda_{2a,S} = \frac{1}{2} \left[\alpha_{2S} + \gamma_{2S} + \sqrt{(\alpha_{2S} - \gamma_{2S})^2 + 4(n-2)\beta_{2S}^2} \right],\quad (\text{B26a})$$

$$\lambda_{2b,S} = \frac{1}{2} \left[\alpha_{2S} + \gamma_{2S} - \sqrt{(\alpha_{2S} - \gamma_{2S})^2 + 4(n-2)\beta_{2S}^2} \right].\quad (\text{B26b})$$

Analogous results are obtained for the A and “+” subspaces. The spin-component degeneracies for the S , A and “+” subspaces are 1, $m-1$ and $m(m-1)/2$ respectively. These have to be multiplied by the degeneracy from the replica sector which is $n-1$.

3. The “−” Spin-Component Sector

The spin-component degeneracy for these eigenvalues is $\frac{1}{2}m(m-1)$.

The blocks with $(\alpha\beta)$ ($\beta \neq \alpha$) and $(\alpha\alpha)$ decouple. The reason is that²² $S_1 = S_2$ and $T_1 = T_2$ which follows from the symmetry properties of the expression for the matrix elements in Eq. (B1) and the definitions in Eqs. (B3) and (B6). It then follows from Eq. (B8) that $S_- = T_- = 0$ and so, from Eq. (B3), the matrix B_- vanishes.

a. The $(\alpha\alpha)$ Subspace

We can easily obtain the “−” eigenvalue which lies entirely within the $(\alpha\alpha)$ subspace, since²² $V_1 = V_2$, so $V_- = 0$ and hence C_- is a diagonal $n \times n$ matrix with

the constant value $U_- = U_1 - U_2$ on the diagonal. Hence there is an eigenvalue

$$\lambda_{4-} = U_- = U_1 - U_2, \quad (\text{B27})$$

with total degeneracy $\frac{1}{2}nm(m-1)$. It has no analogue in the other spin-component sectors.

b. The $(\alpha\beta)$ Subspace, $(\alpha \neq \beta)$

Now we consider the “ $-$ ” eigenvalues which lie entirely within the $(\alpha\beta)$ ($\beta \neq \alpha$) subspace of dimension $n(n-1)/2$. We find that there are two distinct eigenvalues (for general n).

We ask if there are eigenvalues analogous to $\lambda_1, \lambda_2, \lambda_3$ that we found in Appendix A for the Ising case. The eigenvector corresponding to λ_1 for the Ising case has all components equal, since the sum along all rows and columns of the matrix is the same. However, this is not the case here because the number of times $-Q$ occurs is therefore different for different rows or columns. Hence there is no eigenvalue analogous to λ_1 .

Each of the $n-1$ eigenvectors for λ_2 for the Ising case singled out a particular replica. For example, picking out out replica ϵ , the coefficients of (i) $(\epsilon\alpha)$, and (ii) $(\alpha\beta)$ in which neither α nor β equal to ϵ , would be different, see Eq. (A8). There is a similar eigenvector here in which the type (ii) components vanish and the type (i) components are no longer all equal but have value -1 for α less than the special replica (ϵ in our example), and $+1$ for α greater than the special replica. By inspection this has eigenvalue

$$\lambda_{2-} = P + (n-2)Q. \quad (\text{B28})$$

There are n ways to pick the special replica, but the resulting n eigenvectors sum to zero since, in the sum, each element appears twice, once with a plus sign and once with a minus sign. In other words, there is one linear relation between the eigenvectors, so the replica degeneracy of λ_2 is actually $n-1$ rather than n .

For the Ising case, each of the eigenvectors for λ_3 picked out 2 replicas. For example, picking out replicas α and β , the coefficients of (i) $(\alpha\beta)$, (ii) $(\alpha\gamma)$ and $(\beta\gamma)$, and (iii) $(\gamma\delta)$ (all replicas with a different label assumed different) would be different, see Eq. (A13). There are eigenvectors like this here, in which type (iii) components are zero, type (ii) components $(\alpha\gamma)$ have value 1 if $\gamma < \alpha < \beta$ and -1 otherwise, and the type (i) component has value $n-2$. For example, for $n=4$ there are 6 such vectors (not all independent, see below) which are

$$\begin{aligned} \vec{e}_{(12)} &= (2, -1, -1, 1, 1, 0), \\ \vec{e}_{(13)} &= (-1, 2, -1, -1, 0, 1), \\ \vec{e}_{(14)} &= (-1, -1, 2, 0, -1, -1), \\ \vec{e}_{(23)} &= (1, -1, 0, 2, -1, 1), \\ \vec{e}_{(24)} &= (1, 0, -1, -1, 2, -1), \\ \vec{e}_{(34)} &= (0, 1, -1, 1, -1, 2). \end{aligned} \quad (\text{B29})$$

By inspection these have eigenvalue

$$\lambda_{3-} = P - 2Q. \quad (\text{B30})$$

One can also see from the above vectors, which are for $n=4$, that $\vec{e}_{(14)}$ can be expressed as a linear combination of the $\vec{e}_{(1\alpha)}$ for $\alpha < 4$, and similarly $\vec{e}_{(24)}$ can be expressed as a linear combination of the $\vec{e}_{(2\alpha)}$ for $\alpha < 4$, and the same for $\vec{e}_{(34)}$. Hence the last replica can be eliminated, so the number of *linearly independent* vectors, which is the replica degeneracy of λ_{3-} , is $\frac{1}{2}(n-1)(n-2)$.

Hence, including both λ_{2-} and λ_{3-} , we have found all $\frac{1}{2}n(n-1)$ eigenvalues and eigenvectors in the replica sector.

4. Summary of Eigenvalues

The eigenvalues, along with their degeneracies, are summarized in Table III, in which the α 's, β 's and γ 's are defined by

$$\begin{aligned} \alpha_{1S} &= [P_L + (m-1)P_T] + 2(n-2)[Q_L + (m-1)Q_T] \\ &\quad + \frac{1}{2}(n-2)(n-3)[R_L + (m-1)R_T], \end{aligned} \quad (\text{B31a})$$

$$\beta_{1S} = 2[S_L + (m-1)S_T] + (n-2)[T_L + (m-1)T_T], \quad (\text{B31b})$$

$$\gamma_{1S} = [U_L + (m-1)U_T] + (n-1)[V_L + (m-1)V_T], \quad (\text{B31c})$$

$$\begin{aligned} \alpha_{1A} &= (P_L - P_T) + 2(n-2)(Q_L - Q_T) \\ &\quad + \frac{1}{2}(n-2)(n-3)(R_L - R_T), \end{aligned} \quad (\text{B31d})$$

$$\beta_{1A} = 2(S_L - S_T) + (n-2)(T_L - T_T), \quad (\text{B31e})$$

$$\gamma_{1A} = (U_L - U_T) + (n-1)(V_L - V_T), \quad (\text{B31f})$$

$$\begin{aligned} \alpha_{1+} &= (P_1 + P_2) + 2(n-2)(Q_1 + Q_2) \\ &\quad + \frac{1}{2}(n-2)(n-3)(R_1 + R_2), \end{aligned} \quad (\text{B31g})$$

$$\beta_{1+} = 2(S_1 + S_2) + (n-2)(T_1 + T_2), \quad (\text{B31h})$$

$$\gamma_{1+} = (U_1 + U_2) + (n-1)(V_1 + V_2), \quad (\text{B31i})$$

$$\begin{aligned} \alpha_{2S} &= [P_L + (m-1)P_T] + (n-4)[Q_L + (m-1)Q_T] \\ &\quad - (n-3)[R_L + (m-1)R_T], \end{aligned} \quad (\text{B31j})$$

$$\beta_{2S} = [S_L + (m-1)S_T] - [T_L + (m-1)T_T], \quad (\text{B31k})$$

$$\gamma_{2S} = [U_L + (m-1)U_T] - [V_L + (m-1)V_T], \quad (\text{B31l})$$

$$\begin{aligned} \alpha_{2A} &= (P_L - P_T) + (n-4)(Q_L - Q_T) \\ &\quad - (n-3)(R_L - R_T), \end{aligned} \quad (\text{B31m})$$

$$\beta_{2A} = (S_L - S_T) - (T_L - T_T), \quad (\text{B31n})$$

$$\gamma_{2A} = (U_L - U_T) - (V_L - V_T), \quad (\text{B31o})$$

$$\begin{aligned} \alpha_{2+} &= (P_1 + P_2) + (n-4)(Q_1 + Q_2) \\ &\quad - (n-3)(R_1 + R_2), \end{aligned} \quad (\text{B31p})$$

$$\beta_{2+} = (S_1 + S_2) - (T_1 + T_2), \quad (\text{B31q})$$

$$\gamma_{2+} = (U_1 + U_2) - (V_1 + V_2). \quad (\text{B31r})$$

Eigenvalue			replica degeneracy	spin-component degeneracy
$\lambda_{1a,S}$	$\frac{1}{2}$	$\alpha_{1S} + \gamma_{1S} + \sqrt{(\alpha_{1S} - \gamma_{1S})^2 + 2(n-1)\beta_{1S}^2}$	1	1
$\lambda_{1b,S}$	$\frac{1}{2}$	$\alpha_{1S} + \gamma_{1S} - \sqrt{(\alpha_{1S} - \gamma_{1S})^2 + 2(n-1)\beta_{1S}^2}$	1	1
$\lambda_{2a,S}$	$\frac{1}{2}$	$\alpha_{2S} + \gamma_{2S} + \sqrt{(\alpha_{2S} - \gamma_{2S})^2 + 4(n-2)\beta_{2S}^2}$	$n-1$	1
$\lambda_{2b,S}$	$\frac{1}{2}$	$\alpha_{2S} + \gamma_{2S} - \sqrt{(\alpha_{2S} - \gamma_{2S})^2 + 4(n-2)\beta_{2S}^2}$	$n-1$	1
λ_{3S}	$[P_L + (m-1)P_T] - 2[Q_L + (m-1)Q_T] + [R_L + (m-1)R_T]$		$\frac{1}{2}n(n-3)$	1
$\lambda_{1a,A}$	$\frac{1}{2}$	$\alpha_{1A} + \gamma_{1A} + \sqrt{(\alpha_{1A} - \gamma_{1A})^2 + 2(n-1)\beta_{1A}^2}$	1	$m-1$
$\lambda_{1b,A}$	$\frac{1}{2}$	$\alpha_{1A} + \gamma_{1A} - \sqrt{(\alpha_{1A} - \gamma_{1A})^2 + 2(n-1)\beta_{1A}^2}$	1	$m-1$
$\lambda_{2a,A}$	$\frac{1}{2}$	$\alpha_{2A} + \gamma_{2A} + \sqrt{(\alpha_{2A} - \gamma_{2A})^2 + 4(n-2)\beta_{2A}^2}$	$n-1$	$m-1$
$\lambda_{2b,A}$	$\frac{1}{2}$	$\alpha_{2A} + \gamma_{2A} - \sqrt{(\alpha_{2A} - \gamma_{2A})^2 + 4(n-2)\beta_{2A}^2}$	$n-1$	$m-1$
λ_{3A}	$(P_L - P_T) - 2(Q_L - Q_T) + (R_L - R_T)$		$\frac{1}{2}n(n-3)$	$m-1$
$\lambda_{1a,+}$	$\frac{1}{2}$	$\alpha_{1+} + \gamma_{1+} + \sqrt{(\alpha_{1+} - \gamma_{1+})^2 + 2(n-1)\beta_{1+}^2}$	1	$\frac{1}{2}m(m-1)$
$\lambda_{1b,+}$	$\frac{1}{2}$	$\alpha_{1+} + \gamma_{1+} - \sqrt{(\alpha_{1+} - \gamma_{1+})^2 + 2(n-1)\beta_{1+}^2}$	1	$\frac{1}{2}m(m-1)$
$\lambda_{2a,+}$	$\frac{1}{2}$	$\alpha_{2+} + \gamma_{2+} + \sqrt{(\alpha_{2+} - \gamma_{2+})^2 + 4(n-2)\beta_{2+}^2}$	$n-1$	$\frac{1}{2}m(m-1)$
$\lambda_{2b,+}$	$\frac{1}{2}$	$\alpha_{2+} + \gamma_{2+} - \sqrt{(\alpha_{2+} - \gamma_{2+})^2 + 4(n-2)\beta_{2+}^2}$	$n-1$	$\frac{1}{2}m(m-1)$
λ_{3+}	$(P_1 + P_2) - 2(Q_1 + Q_2) + (R_1 + R_2)$		$\frac{1}{2}n(n-3)$	$\frac{1}{2}m(m-1)$
λ_{2-}	$(P_1 - P_2) + (n-2)(Q_1 - Q_2)$		$n-1$	$\frac{1}{2}m(m-1)$
λ_{3-}	$(P_1 - P_2) - 2(Q_1 - Q_2)$		$\frac{1}{2}(n-1)(n-2)$	$\frac{1}{2}m(m-1)$
λ_{4-}	$U_1 - U_2$		n	$\frac{1}{2}m(m-1)$

TABLE III: Eigenvalues and degeneracies for the m -component model. The total degeneracy for each eigenvalue is the product of the replica degeneracy and the spin-component degeneracy. It is easy to see that the total degeneracy is $\frac{1}{2}n(n+1) \times m^2$, as required.

By symmetry²², $R_1 = R_2, S_1 = S_2, T_1 = T_2, V_1 = V_2$.

There are 18 distinct eigenvalues for arbitrary n . In the limit $n \rightarrow 0$, $\alpha_{1S} = \alpha_{2S}$, $\beta_{1S} = \beta_{2S}$, $\gamma_{1S} = \gamma_{2S}$, $\alpha_{1A} = \alpha_{2A}$, $\beta_{1A} = \beta_{2A}$, $\gamma_{1A} = \gamma_{2A}$, $\alpha_{1+} = \alpha_{2+}$, $\beta_{1+} = \beta_{2+}$, $\gamma_{1+} = \gamma_{2+}$, so $\lambda_{1a,S} = \lambda_{2a,S}$, etc., and also $\lambda_{2-} = \lambda_{3-}$. Hence there are only 11 distinct eigenvalues in the $n \rightarrow 0$ limit.

Most of the results in Table III agree with those in de Almeida's thesis²². However, there are some differences, the most notable of which is that the eigenvalue λ_{3S} , which gives the divergence of the non-linear susceptibility according to Eq. (60), does not appear in Ref. 22. However, we are confident that this eigenvalue is correct and that its change of sign gives the AT instability. We note, for example, that the combination of propagators on the LHS of Eq. (B35) corresponds precisely to that in Eq. (3.5) of Ref. 23.

In the Ising ($m = 1$) limit only the “ S ” eigenvalues survive (the degeneracy of the rest is zero), and we present these results in Table IV. One has²² $S_L = T_L = 0$, so $\beta_{1S} = 0$, and $U_L = 1, V_L = 0$, so $\gamma_{1S} = 1$. Hence the first four eigenvalues are $\alpha_{1S}, 1, \alpha_{2S}$ and 1. The two eigenvalues that are equal to 1 involve fluctuations of the $q_{\alpha\alpha}$ which couple to $(S_\alpha)^2$, a constant, (so there is no actual coupling to the spins). Hence these eigenvalues are trivial. The remaining three eigenvalues are just those of the original AT paper², see Table IV.

Eigenvalue		degeneracy
λ_{1a}	$P_L + 2(n-2)Q_L + \frac{1}{2}(n-2)(n-3)R_L$	1
λ_{1b}	1	1
λ_{2a}	$P_L + (n-4)Q_L - (n-3)R_L$	$n-1$
λ_{2b}	1	$n-1$
λ_{3}	$P_L - 2Q_L + R_L$	$\frac{1}{2}n(n-3)$

TABLE IV: Eigenvalues and degeneracies for the Ising case, $m = 1$.

5. Matrix inverse for vector case

As for the Ising case, we assume that G , the matrix inverse of Z , has the same structure as Z itself. In particular, we define

$$G_{(\alpha\beta),(\alpha\beta)}^{\mu\mu,\mu\mu} = G_{1L}, \quad G_{(\alpha\beta),(\alpha\beta)}^{\mu\mu,\nu\nu} = G_{1T} \quad (\mu \neq \nu), \quad (\text{B32a})$$

$$G_{(\alpha\beta),(\alpha\gamma)}^{\mu\mu,\mu\mu} = G_{2L}, \quad G_{(\alpha\beta),(\alpha\gamma)}^{\mu\mu,\nu\nu} = G_{2T} \quad (\mu \neq \nu), \quad (\text{B32b})$$

$$G_{(\alpha\beta),(\gamma\delta)}^{\mu\mu,\mu\mu} = G_{3L}, \quad G_{(\alpha\beta),(\gamma\delta)}^{\mu\mu,\nu\nu} = G_{3T} \quad (\mu \neq \nu), \quad (\text{B32c})$$

$$G_{(\alpha\beta),(\alpha\alpha)}^{\mu\mu,\mu\mu} = G_{4L}, \quad G_{(\alpha\beta),(\alpha\alpha)}^{\mu\mu,\nu\nu} = G_{4T} \quad (\mu \neq \nu), \quad (\text{B32d})$$

$$G_{(\alpha\beta),(\gamma\gamma)}^{\mu\mu,\mu\mu} = G_{5L}, \quad G_{(\alpha\beta),(\gamma\gamma)}^{\mu\mu,\nu\nu} = G_{5T} \quad (\mu \neq \nu), \quad (\text{B32e})$$

where α, β, γ and δ are all different. Considering various matrix elements of both sides of $ZG = I$ we get

$$P_L G_{1L} + (m-1)P_T G_{1T} + 2(n-2)(Q_L G_{2L} + (m-1)Q_T G_{2T}) + \frac{1}{2}(n-2)(n-3)(R_L G_{3L} + (m-1)R_T G_{3T}) + 2(S_L G_{4L} + (m-1)S_T G_{4T}) + (n-2)(T_L G_{5L} + (m-1)T_T G_{5T}) = 1 \quad (\text{B33a})$$

$$P_L G_{1T} + P_T G_{1L} + (m-2)P_T G_{1T} + 2(n-2)(Q_L G_{2T} + Q_T G_{2L} + (m-2)Q_T G_{2T}) + \frac{1}{2}(n-2)(n-3)(R_L G_{3T} + R_T G_{3L} + (m-2)R_T G_{3T}) + 2(S_L G_{4T} + S_T G_{4L} + (m-2)S_T G_{4T}) + (n-2)(T_L G_{5T} + T_T G_{5L} + (m-2)T_T G_{5T}) = 0 \quad (\text{B33b})$$

$$Q_L G_{1L} + (m-1)Q_T G_{1T} + (P_L + (n-2)Q_L + (n-3)R_L)G_{2L} + (m-1)(P_T + (n-2)Q_T + (n-3)R_T)G_{2T} + ((n-3)Q_L + \frac{1}{2}(n-3)(n-4)R_L)G_{3L} + (m-1)((n-3)Q_T + \frac{1}{2}(n-3)(n-4)R_T)G_{3T} + 2(S_L G_{4L} + (m-1)S_T G_{4T}) + (n-2)(T_L G_{5L} + (m-1)T_T G_{5T}) = 0 \quad (\text{B33c})$$

$$Q_L G_{1T} + Q_T G_{1L} + (m-2)Q_T G_{1T} + (P_L + (n-2)Q_L + (n-3)R_L)G_{2T} + (P_T + (n-2)Q_T + (n-3)R_T)G_{2L} + (m-2)(P_T + (n-2)Q_T + (n-3)R_T)G_{2T} + ((n-3)Q_L + \frac{1}{2}(n-3)(n-4)R_L)G_{3T} + ((n-3)Q_T + \frac{1}{2}(n-3)(n-4)R_T)G_{3L} + (m-2)((n-3)Q_T + \frac{1}{2}(n-3)(n-4)R_T)G_{3T} + 2(S_L G_{4T} + S_T G_{4L} + (m-2)S_T G_{4T}) + (n-2)(T_L G_{5T} + T_T G_{5L} + (m-2)T_T G_{5T}) = 0 \quad (\text{B33d})$$

$$R_L G_{1L} + (m-1)R_T G_{1T} + (4Q_L + 2(n-4)R_L)G_{2L} + (m-1)(4Q_T + 2(n-4)R_T)G_{2T} + (P_L + 2(n-4)Q_L + \frac{1}{2}(n-4)(n-5)R_L)G_{3L} + (m-1)(P_T + 2(n-4)Q_T + \frac{1}{2}(n-4)(n-5)R_T)G_{3T} + 2(S_L G_{4L} + (m-1)S_T G_{4T}) + (n-2)(T_L G_{5L} + (m-1)T_T G_{5T}) = 0 \quad (\text{B33e})$$

$$R_L G_{1T} + R_T G_{1L} + (m-2)R_T G_{1T} + (4Q_L + 2(n-4)R_L)G_{2T} + (4Q_T + 2(n-4)R_T)G_{2L} + (m-2)(4Q_T + 2(n-4)R_T)G_{2T} + (P_L + 2(n-4)Q_L + \frac{1}{2}(n-4)(n-5)R_L)G_{3T} + (P_T + 2(n-4)Q_T + \frac{1}{2}(n-4)(n-5)R_T)G_{3L} + (m-2)(P_T + 2(n-4)Q_T + \frac{1}{2}(n-4)(n-5)R_T)G_{3T} + 2(S_L G_{4T} + S_T G_{4L} + (m-2)S_T G_{4T}) + (n-2)(T_L G_{5T} + T_T G_{5L} + (m-2)T_T G_{5T}) = 0 \quad (\text{B33f})$$

Forming appropriate linear combinations of Eqs. (B33) gives

$$\left([G_{1L} + (m-1)G_{1T}] - 2[G_{2L} + (m-1)G_{2T}] + [G_{3L} + (m-1)G_{3T}]\right) \times \left([P_L + (m-1)P_T] - 2[Q_L + (m-1)Q_T] + [R_L + (m-1)R_T]\right) = 1. \quad (\text{B34})$$

so the “replicon” propagator is given by

$$G_r \equiv [G_{1L} + (m-1)G_{1T}] - 2[G_{2L} + (m-1)G_{2T}] + [G_{3L} + (m-1)G_{3T}] = \left([P_L + (m-1)P_T] - 2[Q_L + (m-1)Q_T] + [R_L + (m-1)R_T]\right)^{-1}. \quad (\text{B35})$$

Appendix C: Averages over spin directions

To evaluate the spin glass susceptibility we need to compute averages over spin directions. Consider, for example,

$$Z = \int d\Omega_m \exp[\mathbf{H} \cdot \mathbf{e}], \quad (\text{C1})$$

where the integral is over the surface, Ω_m , of a sphere of unit radius, \mathbf{e} is a unit vector whose direction is to be integrated over, and \mathbf{H} is a fixed vector.

Working in polar coordinates, with the polar axis along the direction of the fixed vector \mathbf{H} , the integral in Eq. (C1) can be expressed entirely in terms of the polar angle θ , since $\exp[\mathbf{H} \cdot \mathbf{e}] = \exp[H \cos \theta]$ and $\int d\Omega_m = C_m \int_0^\pi \sin^{m-2} \theta$ for a constant C_m . To deter-

mine C_m we note the following results^{35,36},

$$\Omega_m \equiv \int d\Omega_m = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})}, \quad (\text{C2})$$

$$\int_0^\pi \sin^{m-2} \theta d\theta = \sqrt{\pi} \frac{\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m}{2})}, \quad (\text{C3})$$

where Γ is the usual Gamma function, which gives

$$C_m = \frac{2\pi^{(m-1)/2}}{\Gamma(\frac{m-1}{2})}. \quad (\text{C4})$$

Hence Z can be written as

$$Z = \frac{2\pi^{(m-1)/2}}{\Gamma(\frac{m-1}{2})} \int_0^\pi \exp[H \cos \theta] \sin^{m-2} \theta d\theta. \quad (\text{C5})$$

The integral is given in terms of a modified Bessel function³⁵, and we have

$$Z = (2\pi)^{m/2} \frac{I_{m/2-1}(H)}{H^{m/2-1}}. \quad (\text{C6})$$

Of greater interest are averages of the spins. Consider first

$$\langle S_\mu \rangle = m^{1/2} \langle e_\mu \rangle, \quad (\text{C7})$$

$$= m^{1/2} \frac{1}{Z} \frac{\partial Z}{\partial H^\mu}, \quad (\text{C8})$$

$$= m^{1/2} \frac{1}{Z} \frac{H^\mu}{H} \frac{\partial Z}{\partial H}. \quad (\text{C9})$$

Using³⁵

$$\frac{d}{dx} \left[\frac{I_{m/2-1}(x)}{x^{m/2-1}} \right] = \frac{I_{m/2}(x)}{x^{m/2-1}}, \quad (\text{C10})$$

we get

$$\langle S_\mu \rangle = m^{1/2} \frac{H^\mu}{H} \frac{I_{m/2}(H)}{I_{m/2-1}(H)}. \quad (\text{C11})$$

We shall also need

$$\langle S_\mu S_\nu \rangle = m \frac{1}{Z} \frac{H^\mu}{H} \frac{\partial}{\partial H} \left(\frac{H^\nu}{H} \frac{\partial Z}{\partial H} \right), \quad (\text{C12})$$

$$= \frac{m}{I_{m/2-1}(H)} \left[\delta_{\mu\nu} \frac{I_{m/2}(H)}{H} + \frac{H_\mu H_\nu}{H^2} I_{m/2+1}(H) \right], \quad (\text{C13})$$

in which we again used Eq. (C10).

To apply these results, we note that, in the presence of an external random field, the replica symmetric solution predicts that \mathbf{H} is given by

$$\mathbf{H} = \beta m^{1/2} (J^2 q + h_r^2) \mathbf{z}. \quad (\text{C14})$$

where each component of the variable \mathbf{z} is a Gaussian random variable with zero mean and standard deviation unity. To see this, compare Eq. (53) with Eq. (C1)

and note that the spins are of length $m^{1/2}$ according to Eq. (2). Hence each component of \mathbf{H} has zero mean and standard deviation given by

$$\Delta = \beta m^{1/2} (J^2 q + h_r^2)^{1/2}. \quad (\text{C15})$$

As for the Ising case, we denote averages over \mathbf{H} , or equivalently over \mathbf{z} (\mathbf{H} and \mathbf{z} are related by Eq. (C14)), by $[\cdots]_z$ and so, for example, in situations which only involve the magnitude of \mathbf{H} , we have

$$\begin{aligned} [f(H)]_z &= \int_{-\infty}^{\infty} \left(\prod_{\mu} \frac{dH_{\mu}}{(2\pi)^{1/2} \Delta} \right) e^{-\sum_{\mu} H_{\mu}^2 / 2\Delta^2} f(H) dH, \\ &= \frac{\Omega_m}{(2\pi)^{m/2} \Delta^m} \int_0^{\infty} H^{m-1} \exp\left(-\frac{H^2}{2\Delta^2}\right) f(H) dH, \\ &= \frac{2^{1-m/2}}{\Delta^m \Gamma(\frac{m}{2})} \int_0^{\infty} H^{m-1} \exp\left(-\frac{H^2}{2\Delta^2}\right) f(H) dH, \end{aligned} \quad (\text{C16})$$

$$= \frac{2^{1-m/2}}{\Gamma(\frac{m}{2})} \int_0^{\infty} z^{m-1} e^{-z^2/2} f(\Delta z) dz, \quad (\text{C17})$$

where we used the result for Ω_m in Eq. (C2), and Δ is given by Eq. (C15).

Using these results we now calculate the spin glass order parameter q , which is given by

$$\begin{aligned} q &= \frac{1}{m} \left[\sum_{\mu=1}^m \langle S_{\mu} \rangle^2 \right]_z \\ &= \sum_{\mu=1}^m \frac{(H^{\mu})^2}{H^2} \left[\left(\frac{I_{m/2}(H)}{I_{m/2-1}(H)} \right)^2 \right]_z \\ &= \left[\left(\frac{I_{m/2}(H)}{I_{m/2-1}(H)} \right)^2 \right]_z \\ &= \frac{2^{1-m/2}}{\Delta^m \Gamma(\frac{m}{2})} \int_0^{\infty} dH H^{m-1} \exp\left(-\frac{H^2}{2\Delta^2}\right) \left(\frac{I_{m/2}(H)}{I_{m/2-1}(H)} \right)^2 \\ &= \frac{2^{1-m/2}}{\Gamma(\frac{m}{2})} \int_0^{\infty} dz z^{m-1} e^{-z^2/2} \left(\frac{I_{m/2}(\Delta z)}{I_{m/2-1}(\Delta z)} \right)^2, \end{aligned} \quad (\text{C18})$$

where we used Eq. (C11). Equation (C18), with Δ given by Eq. (C15), is the self-consistent equation which determines q . As an example, for $m = 1$, $I_{m/2}(H)/I_{m/2-1}(H) = \tanh(H) = \tanh(\Delta z)$, and we recover the result for q in Eq. (29). For general m , expanding the Bessel functions for small argument³⁵, we get

$$\begin{aligned} q &= \left[\frac{1}{m^2} H^2 - \frac{2}{m^3(m+2)} H^4 + \right. \\ &\quad \left. \frac{5m+12}{m^4(m+2)^2(m+4)} H^6 + O(H^8) \right]_z \end{aligned} \quad (\text{C19})$$

If we do the Gaussian integrals, set $h_r = 0$, and solve for q , we find

$$q = t + \frac{1}{m+2} t^2 + O(t^3), \quad (h_r = 0), \quad (\text{C20})$$

where t , the reduced temperature, is given by $t = (T_c - T)/T_c$, and the zero field transition temperature is $T_c = J$, see Eq. (5).

Our main goal is to compute the eigenvalue λ_{3S} since this determines the spin glass susceptibility, the divergence of which indicates the location of the AT line. From Eqs. (B20a), (A2), (B5), (B1) and (B2), we find the fairly simple expression

$$\lambda_{3S} = 1 - (\beta J)^2 \frac{1}{m} \sum_{\mu, \nu} [\langle S_\mu S_\nu \rangle^2 - 2\langle S_\mu S_\nu \rangle \langle S_\mu \rangle \langle S_\nu \rangle + \langle S_\mu \rangle^2 \langle S_\nu \rangle^2]_z, \quad (\text{C21})$$

which is instructive to write in the following form

$$\lambda_{3S} = 1 - (\beta J)^2 \chi_{SG}^0, \quad (\text{C22})$$

where χ_{SG}^0 is a single-site spin glass susceptibility,

$$\chi_{SG}^0 = \frac{1}{m} \sum_{\mu, \nu} \left[\left(\langle S_\mu S_\nu \rangle - \langle S_\mu \rangle \langle S_\nu \rangle \right)^2 \right]_z. \quad (\text{C23})$$

Evaluating the spin averages in Eq. (C23) using Eqs. (C11) and (C13) gives

$$\chi_{SG}^0 = m \left[\frac{1}{I_{m/2-1}^2(H)} \left\{ \frac{m}{H^2} I_{m/2}^2(H) + \frac{2}{H} I_{m/2}(H) I_{m/2+1}(H) + I_{m/2+1}^2(H) \right\} - \frac{2}{I_{m/2-1}^3(H)} \left\{ \frac{1}{H} I_{m/2}^3(H) + I_{m/2}^2(H) I_{m/2+1}(H) \right\} + \left\{ \frac{I_{m/2}(H)}{I_{m/2-1}(H)} \right\}^4 \right]_z. \quad (\text{C24})$$

We recall that the average over H is evaluated according to Eq. (C16). For the Ising case, $m = 1$, Eq. (C24) simplifies to

$$\chi_{SG}^0 = [1 - 2 \tanh^2 H + \tanh^4 H]_z, \quad (\text{C25})$$

in agreement with Eq. (45). For the Heisenberg case, $m = 3$, Eq. (C24) becomes

$$\chi_{SG}^0 = 3 \left[\frac{3 + 2H^2 - 4H \coth(H)}{H^4} + \frac{1}{\sinh^4(H)} \right]_z, \quad (\text{C26})$$

which, together with Eqs. (C16) and (C22), gives λ_{3S} . Equations (C24) and (C26) appear to be a new results. Expanding the Bessel functions for small H gives

$$\chi_{SG}^0 = \left[1 - \frac{2}{m^2} H^2 + \frac{5m+16}{m^3(m+2)^2} H^4 + O(H^6) \right]_z. \quad (\text{C27})$$

Let us evaluate q and λ_{3S} near $T = T_c (= J)$, the zero field transition temperature, and for small h_r . Using Eqs. (C19) and (C27), and doing the Gaussian integrals, we find

$$q = \tilde{\Delta}^2 - 2\tilde{\Delta}^4 + \frac{5m+12}{m+2} \tilde{\Delta}^6 + \dots, \quad (\text{C28})$$

$$\lambda_{3S} = 1 - (\beta J)^2 \left[1 - 2\tilde{\Delta}^2 + \frac{5m+16}{m+2} \tilde{\Delta}^4 + \dots \right], \quad (\text{C29})$$

where

$$\tilde{\Delta}^2 \equiv \frac{\Delta^2}{m} = \beta^2 (J^2 q + h_r^2). \quad (\text{C30})$$

Combining Eqs. (C29) and (C28), and assuming

$$h_r \ll t \equiv (T_c - T)/T_c \ll 1, \quad (\text{C31})$$

which will be valid at and below the AT line near T_c , we get

$$\lambda_{3S} = \left(\frac{h_r}{J} \right)^2 \frac{1}{q} - \frac{4}{m+2} q^2. \quad (\text{C32})$$

In the limits of Eq. (C31), we have $q = t + O(t^2)$, see Eqs. (C20) and (C35), and so

$$\lambda_{3S} = \left(\frac{h_r}{J} \right)^2 \frac{1}{t} - \frac{4}{m+2} t^2, \quad (h_r \ll t), \quad (\text{C33})$$

which changes sign for

$$\left(\frac{h_r}{J} \right)^2 = \frac{4}{m+2} t^3. \quad (\text{C34})$$

Equation (C34) gives the location of the AT line for an m -component spin glass near the zero field transition. The replica symmetric solution is unstable at lower temperatures and fields since $\lambda_{3S} < 0$ in that region according to Eq. (C33). Note that Eq. (C34) correctly gives the AT result that $h_r^2 = (4/3) t^3$ for $m = 1$ (This is a valid comparison even though AT used a uniform field since, to lowest order in t , the location of the AT line in the Ising case is the same⁷ for random and uniform fields.) On the AT line we find that the spin glass order parameter is given by

$$q = t + \frac{3}{m+2} t^2 + O(t^3), \quad (\text{on AT line}). \quad (\text{C35})$$

In the opposite limit, $T \rightarrow 0$, we find, using properties of the Bessel functions, that

$$\frac{h_r(T \rightarrow 0)}{J} = \frac{1}{\sqrt{m-2}} \quad (m > 2), \quad (\text{C36})$$

while $h_r(T \rightarrow 0)$ diverges for $m \leq 2$. For the Ising case,

we get

$$\frac{h_r(T \rightarrow 0)}{J} = \sqrt{\frac{8}{9\pi}} \frac{J}{T} \quad (m = 1), \quad (\text{C37})$$

in agreement with Bray⁷.

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